Intro to Math Thinking **Name:**
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Sets Functions Worksheet

1. Function compositions are noncommutative. Informally, **noncommutative** means that *order matters.*

   More formally if $\star$ is some binary operation (two inputs and one output, an example can be addition or multiplication and so on) on a set, and x and y are elements of that set, then noncommutative means that $x \star y$ does not necessarily equal $y * x$.

   Most common operations, such as *addition* and *multiplication* of numbers are *commutative.* For instance $4 \cdot 3 = 12 = 3 \cdot 4$ and $2 + 3 = 5 = 3 + 2$.

   If $f(x)$ and $g(x)$ are functions, then usually $(f \circ g)(x) \neq (g \circ f)(x)$. This can also be written as $f(g(x)) \neq g(f(x))$.

   For example, suppose $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ and $g(x) = x + 1$.

   Then can you find the following:

   (a) First apply g and then apply f to the output of g, ie,

   $$f \circ g(x) = f(g(x)) =?$$

   (b) Now do the opposite, apply f first and then apply g to the output of f, ie,

   $$g \circ f(x) = g(f(x)) =?$$

   (c) Are they the same?

2. Counting Injective, Bijective, Surjective functions. Recall that a function is

- *injective* or *one-one*: if two different inputs do not give the same output. In other words, $x \neq y \to f(x) \neq f(y)$. An example of a function that is **not injective** is $f(x) = x^2$ since for example $-2 \neq 2$ but $f(-2) = 4 = f(2)$.
- *surjective* or *onto*: If every element in the codomain has at least one preimage. In other words, for $f : A \to B$ if $b \in B$, then there exists (at least one) $a \in A$ such that $f(a) = b$. An example of a function that is **not surjective** is the following $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$. Then $-4 \in \mathbb{R}$ is in the codomain but it does not have any preimage. In other words, there is no $x \in \mathbb{R}$ such that $f(x) = -4$.
- *bijective*: If its both injective and surjective. Note that this means that the for $f : A \to B$ bijective, we must have that $|A| = |B|$ ie A and B must have the same number of elements. When such a scenario holds, alternatively we use the terminology A and B are in bijection and f is a bijection from A to B.

Try the following exercise:

(a) Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. How many injective functions are there from A to B?

(b) Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. How many surjective functions are there from B to A?

(c) Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3\}$. How many bijective functions are there from A to B?