Intro to Math for CS
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Group Project Ideas

Instructions:

- During the last two days (July 27 and July 28), each group must present a proof of a mathematical result, or do a programming assignment and demonstrate it.
- Each group will have 15 minutes to present their project.
- By the end of Week 2 (July 14) please let us know which project you will be doing. Please ask us questions if you are stuck or need help on deciding on a project.
- You can also come up with your own project ideas and let us know by Week 2.
- Note that this is a group activity, you can communicate with your group during the activities time. We will have dedicated activities sessions where you can work with your group members.

Project Ideas

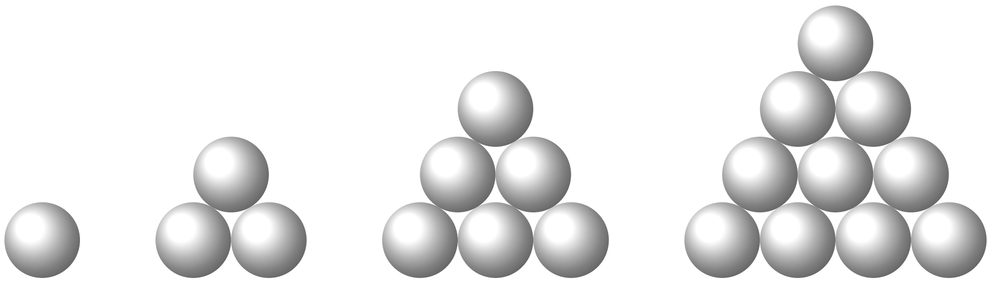
You may present a proof of a mathematical result. Here are a few project ideas. The project will involve presenting a complete proof of a mathematical fact.

1. We all know that prime numbers are infinite. Can you come up with 2 different proofs of why there are infinitely many primes.
2. You may have heard about irrational numbers. Show that the square root of 2 is irrational. Furthermore, show that the square root of any prime number p is irrational.
3. Consider the following exercise:

$$\begin{aligned} t_1 &= 1 = 1 \\ t_2 &= 1 + 2 = 3 \\ t_3 &= 1 + 2 + 3 = 6 \\ t + 4 &= 1 + 2 + 3 + 4 = 10 \\ \vdots + \vdots + \vdots + \vdots &= \vdots \\ t_n = 1 + \cdots + \cdots + (n-1) + n &= \frac{n(n+1)}{2} \end{aligned}$$

   Why is it true?

   These numbers are also called *triangular numbers* denoted t_n because they count the number of balls inside an equilateral triangle.



Similarly, can you come up with a formula for the following and prove it?

$$
\begin{aligned}
1^2 &= 1 \\
1^2 + 2^2 &= 5 \\
1^2 + 2^2 + 3^3 &= 14 \\
\vdots + \vdots + \vdots &= \vdots \\
1^2 + \cdots + \cdots + (n-1)^2 + n^2 &= ??
\end{aligned}
$$

4. The famous fibonacci sequence involves the following:

   - Start with two numbers $F_1 = 1$ and $F_2 = 1$
   - The third number F_3 is the sum of the first and second giving $F_3 = F_2 + F_1 = 1 + 1 = 2$.
   - Similarly, the fourth number is the sum of the third and the second $F_4 = F_3 + F_2 = 2 + 1 = 3$.
   - We keep repeating, ie, the n-th number F_n is the sum of the previous two $F_n = F_{n-1} + F_{n-2}$

   Print out the first 10 fibonacci numbers. How large are the numbers?

   Can you write a program that works this out for you? Can you prove the following:

   (a)

   $$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$$

   (b)

   $$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

   (c)

   $$F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$$

   (d)

   $$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

   (e) Is there something that is interesting that you observe? Explain your findings with a proof or some reasoning as to why.

5. Suppose there are n people and only k movie tickets. Say for example there are 6 people and 4 movie tickets. How many ways can they go to the movie? For example

   - one way would be when Person 1, 2, 3,4 goes and Person 5,6 do not go.
   - another way would be when person 1,2,3,5 go to the movie and person 4,6 does not.

   Can you find all the number of ways this can be done?

   For example when $n = 3$ and $k = 2$, then it can be done in 3 ways:

   (a) Person 1, 2 go , 3 does not.

   (b) Person 2, 3 go, 1 does not.

   (c) Person 1, 3 go, 2 does not.

   Remember that we are interested in only the number of ways it can be done, and not the entire description of how. For the example above it was 3 ways when we had 3 people and 2 tickets.

   We denote this number by $\binom{n}{k}$.

   Can you prove the following facts:

   (a) $\binom{n}{k} = \binom{n}{n-k}$ This means that the number of ways 10 people can go to a movie with only 4 tickets, is equal to the number of ways 10 people can go to a movie with only 6 tickets. Can you see why?

   (b) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ This says for example that the number of ways 10 people can go to a movie with only 4 tickets, is equal to the number of ways 9 people can go to a movie with 4 tickets plus the number of ways 9 people can go to a movie with 3 tickets. Can you prove why?

   (c) Are there interesting patterns that you observe? Explain.

6. A programming project can be to print out the Pascals triangle which involves the coefficients $\binom{n}{k}$ (read the previous question for an introduction to $\binom{n}{k}$)

$n = 0$							1						
$n = 1$						1		1					
$n = 2$					1		2		1				
$n = 3$				1		3		3		1			
$n = 4$			1		4		6		4		1		
$n = 5$		1		5		10		10		5		1	
$n = 6$	1		6		15		20		15		6		1

   - It starts off with 1 on the first row and
   - 1 1 on the second row.
   - For the third row the first and last terms are 1, but the middle term is obtained by summing the two numbers on the right and left of it in the previous row. For instance $2 = 1+1$
   - For the fourth row we do similarly, $3 = 1 + 2, \quad 3 = 2 + 1$

- We keep repeating.

Questions:

(a) Why does it involve $\binom{n}{k}$?

(b) Do you see why $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Explain.

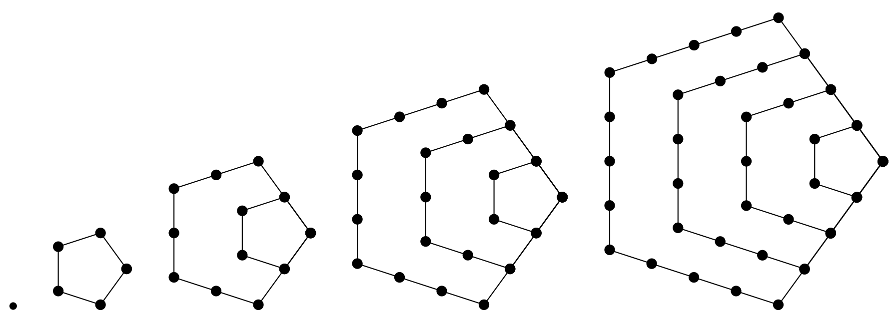
(c) Maybe write a program that takes an input n and prints out the first n rows of the Pascals triangle.

(d) Feel free to investigate these numbers and report interesting patterns or observations that you make.

7. Each of the numbers

$$\begin{aligned} p_1 &= 1 \\ p_2 &= 5 = 1 + 4 \\ p_3 &= 12 = 1 + 4 + 7 \\ p_4 &= 22 = 1 + 4 + 7 + 10 \end{aligned}$$

represents the number of dots that can be arranged evenly in a pentagon



These numbers p_n are known as *pentagonal numbers.*

Can you prove the following:

(a) $p_n = p_{n-1} + (3n - 2)$ for $n \geq 2$.

(b) Can you come up with a general formula for p_n?

(c) Prove that $p_n = t_{n-1} + n^2$ and $p_n = 3t_{n-1} + n = 2t_{n-1} + t_n$ where t_n are the *triangular numbers.*

(d) Can you find some more interesting patterns? Do you observe anything interesting? Report your findings.

8. Write a program in python that plays chess between players. Some things that you can do include:

- The program prints out a representation of a chess board and keeps track of where all the pieces are.

- The player inputs a move for example Nf3 (Knight moves to f3 read about FEN notation on Wikipedia)

- The program checks if the move is legal and updates the board if it is.

9. Project Set Game

   - Write a program in python that lets you play Projective Set. The program prints out a list of cards 1 (R O P) 2 (Y G P) 3 (R O B) 4 (O G B) 5(R O Y B P) 6(Y G) 7(R O G)
   - Then the user inputs a guess, for example 1,2,6
   - The computer then checks if the user's guess forms a set.
   - Feel free to investigate other interesting questions you have.

10. The 4 square theorem says that every natural number can be written as the sum of four squares.

    For example

$$\begin{aligned} 3 &= 1^2 + 1^2 + 1^2 + 0^2 \\ 31 &= 5^2 + 2^2 + 1^2 + 1^2 \\ 310 &= 17^2 + 4^2 + 2^2 + 1^2 \\ &= 16^2 + 7^2 + 2^2 + 1^2 \\ &= 15^2 + 9^2 + 2^2 + 0^2 \\ &= 12^2 + 11^2 + 6^2 + 3^2 \end{aligned}$$

    The theorem was proven by Lagrange in 1770. It is a special case of Fermat's polygonal number theorem.

    (a) Read about the polygonal numbers on Wikipedia `https://en.wikipedia.org/wiki/Polygonal_number`

    (b) Write a program that counts the number of ways to write a number as the sum of four squares.

    (c) Alternatively write a program that lists the numbers which are not a sum of three squares.

    (d) Feel free to investigate other interesting questions you have.

11. Approximating π. Try generating random real numbers between -1 and 1. For example: 0.4567

    A pair of these numbers say $x = 0.345, y = 0.789$ can be thought of as lying inside a square in 2D of side 2.

    Generate around 1000 random points. Which of these lie inside the circle centered at origin of radius 1?

    Can you use this to approximate π?

12. Read about Newton's Root finding method. Write a program in python that finds the roots of a function.

If you want to explore more topics other than these feel free to decide on your own topics and discuss with us during Week 2