Intro to Math for CS **NAME:**
Instructors: Junaid Hasan and Leopold Mayer

Lecture Notes July 6

Tautologies and Contradictions

- A **tautology** is a compound statement that is always true regardless of the truth values of the individual statements subsituted for its statement variables.
- A **contradiction** is a compound statement that is always false regardless of the truth values of the individual statements substituted for its statement variables.

  As an example

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

  Note that $P \vee \neg P$ is a **tautology** while $P \wedge \neg P$ is a **contradiction.**

Predicates and Quantified Statements

Suppose I say the following:

"He is a college student"

This is not a statement because it may be true or false depending on the value of the pronoun *he*.

In grammar, the word predicate refers to the part of a sentence that gives information about the subject. In the sentence

"James is a student at the University of Washington", we have that

- *James* is the subject.
- *is a student at the University of Washington* is the predicate.

Whereas in logic, predicates can be obtained by removing some or all of the nouns from a statement.

In the example above we can remove

- *James* and say
$$x \text{ is a student at the University of Washington.}$$
- *James* and *the University of Washington* and say
$$x \text{ is a student at } y.$$

Therefore we have the following definition:

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

For example consider the following predicate:

$$P(x): \quad x^2 > x$$

It may be **true** and **false.**

- It is *true* for example when $x = 2$ because

$$2^2 = 4 > 2.$$

- It is *false* for example when $x = \frac{1}{2}$ because

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not> \frac{1}{2}.$$

Let us now define **quantifiers**. Let us ask the question: How do I change a predicate (which may be true and false for different values of the variables) into a statement? The answer is by substituting a number.

However, can we do it in another way?

1. An answer is via the "for all" quantifier $\forall$. Suppose we have the following statement

   $$\forall \text{ human beings} x, \quad x \text{ is mortal.}$$

   We have used the symbol x, so you may think of it being a predicate, but using the $\forall$ quantifier makes sure that we are making it a statement, it can either be true or false.

   A **universal statement** is a statement of the form

   $$\forall x, \quad P(x)$$

   If a statement of the above form is false, it means that for some x we must have $P(x)$ to be false. Such an x is called a counterexample.

   An example is

   $$\forall x, \quad x^2 > x$$

   is false because of a counterexample $x = 1$, or $x = \frac{1}{2}$.

2. Another way is to use the "there exists" quantifier $\exists$ As an example

   $$\exists \text{a person } P, \quad \text{such that } P \text{ is a student in Intro to Math for CS}$$

Another example

$$\exists \text{a number } n \text{such that } n^2 = n$$

is true because $1^2 = 1$.

Try the following exercise: Formulate the statements below using $\exists$ and $\forall$.

- All numbers have nonnegative squares.
- We can find at least one positive number whose square is equal to itself.
- All triangles have three sides.
- No dogs have wings.

Universal Conditional Statements

In mathematics one of the most important forms of a statement is

$$\forall \ldots\ldots, \quad \text{if} \ldots\ldots \text{then} \ldots\ldots,$$

or more concretely

$$\forall x, \quad \text{if} \quad P(x) \quad \text{then} \quad Q(x),$$

Write the following statements as a universal conditional

- If a number is greater than 2, its square is greater than 4.
- All bytes have eight bits.
- No fire trucks are green.
- All squares are rectangles.

Negation of Quantified Statements

Consider the statement "All mathematicians wear glasses". What would be its negation?

Many would say "No mathematicians wear glasses", but that is not correct.

Its correct negation is "There is at least one mathematician who does not wear glasses."

The general form of a negation is the following:

If a statement is of the form

$$\forall x \quad , Q(x)$$

then its negation is

$$\exists x \quad \text{such that } \neg Q(x).$$

Similarly lets motivate negation of the quantified statement. Suppose I say “Some are the same”. What is its negation?

Its negation would amount to saying that All cars are unique, for even two cars being same would make the above hold. Its negation would be something like “No cars are the same” or ‘All cars are different”.

If a statement is of the form

$$\exists x \quad \text{such that } Q(x)$$

then its negation is

$$\forall x \quad \neg Q(x).$$

Write negations of the following:

- For all primes p , p is odd.
- There exists a triangle T, such that the sum of its angles equals 200^o.
- No policitians are honest

  A formal answer is the following

  $$\forall \text{ politicians } x, x \text{ is not honest.}$$

  Its formal negation is

  $$\exists \text{a politician } x \text{such that } x \text{ is not honest.}$$

  Its informal negation is

  Some politicians are honest.

- All computer programs are finite.
- Some computer hackers are over 40.
- The number 1357 is divisible by some integer between 1 and 37.

Negation of Universal Conditional Statements

Negation of a statement

$$\forall x, \text{if } P(x), \quad \text{then } Q(x)$$

is the following

$$\exists x, \quad \text{such that } P(x) \text{and } \neg Q(x).$$

Negate the following

- $\forall$people P, if $P$is blond then Phas blue eyes..
- If a computer program has more than 100,000 lines, then it contains a bug.

More things to say For a statement $\forall x$, if $P(x)$ then $Q(x)$.

Contrapositive: $\forall x$, if $\neg Q(x)$, then $\neg P(x)$.

Converse $\forall x$, if $Q(x)$ then $P(x)$.

Inverse $\forall x$, if $\neg P(x)$ then $\neg Q(x)$.

Question: Write the contrapositive, converse and inverse of the following:

If a number is greater than 2, then its square is greater than 4.

The formal version is $\forall n$, if $n > 2$, then $n^2 > 4$.

Negation of Conditional Statement

Lets start with the following exercise:

Using truth tables show the following:

- $P \vee Q \to R \equiv (P \to R) \wedge (Q \to R)$
- $P \to Q \equiv \neg P \vee Q$

  An example of the above is

  If you do not get to work on time, then you are fired, is the same as

  Either you get to work on time or you are fired.

This means that

$$\neg(P \to Q) \equiv (P \wedge \neg Q)$$

- The **contrapositive** of If p then q is

  $$\text{If} \neg q \text{ then } \neg p.$$

  Furthermore, the contrapositive is logically equivalent to the original statement.

  Write contrapositives of the following

  1. If Howard can swim across the lake, then Howard can swim to the island.
  2. If today is Easter, then tomorrow in Monday.

- The **converse** of If p then q is

  $$\text{If } q \text{ then } p.$$

- The inverse is

  $$\text{If } \neg p \text{ then } \neg q.$$

  Write converse and inverse of the following

  1. If Howard can swim across the lake, then Howard can swim to the island.

2. If today is Easter, then tomorrow in Monday.

A conditional and its converse are not logically equivalent.

A conditional and its inverse are not logically equivalent.

Necessary and Sufficient conditions

- R is a **sufficient condition** for S means that

$$R \to S$$

We also say if R then S. Sometimes we also say R only if S.

Because the above means

$$\neg S \to \neg R \quad \equiv \quad R \to S$$

- R is a **necessary condition** for S means that

$$\neg R \to \neg S \quad \equiv \quad S \neg R$$

We also say R if S .