Intro to Math for CS **Name:**
Instructors: Junaid Hasan and Leopold Mayer

Generating Functions Worksheet

1. A certain sequence of numbers $a_0, a_1, \ldots$ satisfies the conditions:

$$a_{n+1} = 2a_n + 1 \qquad (n \geq 0; \quad a_0 = 0)$$

   (a) Compute the first 10 terms of the sequence.

   (b) Does the sequence look familiar? Can you guess a general formula for a_n?

   (c) Instead let us focus on the generating function:

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

   To find $A(x)$, multiply both sides of the recurrence $a_{n+1} = 2a_n + 1$ by x^n and sum over the values of n where the recurrence is valid, namely over $n \geq 0$. Then try to relate these sums to the unknown generating function $A(x)$.

   A useful formula $\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots$

2. A certain sequence of numbers $a_0, a_1, \ldots$ satisfies the conditions:

$$a_{n+1} = 2a_n + n \qquad (n \geq 0; \quad a_0 = 1)$$

(a) Compute the first 10 terms of the sequence.

(b) Does the sequence look familiar? Can you guess a general formula for a_n? This time it may be hard to guess. Therefore lets opt for a generating function approach.

(c) Instead let us focus on the generating function:

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

To find $A(x)$, multiply both sides of the recurrence $a_{n+1} = 2a_n + n$ by x^n and sum over the values of n where the recurrence is valid, namely over $n \geq 0$. Then try to relate these sums to the unknown generating function $A(x)$.

A useful formula $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 \cdots$

3. Suppose

$$A(x) = \frac{1 - 2x + 2x^2}{(1 - x)^2(1 - 2x)} = \frac{A}{(1 - x)^2} + \frac{B}{1 - x} + \frac{C}{1 - 2x}.$$

This is known as a partial fraction decomposition. How to find A, B, C?

(a) First multiply both sides by $(1 - x)^2$ and let $x = 1$, what do you get?

(b) Instead if you multiply by $(1 - 2x)$ and let $x = \frac{1}{2}$ what do you get?

(c) To find the third variable, substitute an easy value, say $x = 0$, what do you get?