Week-6-Homework-Solutions

July 28, 2021

0.1 Problem 1

Let us consider the following problem:

We roll a die and keep track of the sum. * If we roll 1 we **subtract** from the sum, except when the sum is already 0, when we do nothing. * If we roll 2,3,4,5,6 we **add** to the sum. * We stop the moment we reach 8 or higher.

- Example: Say we roll 1,4,1,2,4.

1. The sum is 0, so roll 1 keeps sum 0.
2. The roll 4, adds and the sum is 4.
3. The roll 1, subtracts and the sum is 3.
4. The roll 2, adds and the sum is 5.
5. The roll 4, adds and the sum is 9 and we stop.

1. **Question**: Find the transition matrix?

Answer: The matrix $P = \begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{3}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{4}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

[41]:

```python
import sympy
oneby6 = sympy.Rational(1,6)
P = sympy.Matrix([[oneby6, 0, oneby6, oneby6, oneby6, oneby6, oneby6, 0 , 0],
                  [oneby6, 0, 0, oneby6, oneby6, oneby6, oneby6, oneby6, 0],
                  [0, oneby6, 0, 0, oneby6, oneby6, oneby6, oneby6, oneby6],
                  [0, 0, oneby6, 0, 0, oneby6, oneby6, oneby6, 2*oneby6],
                  [0, 0, 0, oneby6, 0, 0, oneby6, oneby6, 3*oneby6],
                  [0, 0, 0, 0, oneby6, 0, 0, oneby6, 4*oneby6],
                  [0, 0, 0, 0, 0, oneby6, 0, 0, 5*oneby6],
                  [0, 0, 0, 0, 0, 0, oneby6, 0, 5*oneby6],
                  [0, 0, 0, 0, 0, 0, 0, 0, 1]])
P
```

[41]:

$$\begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. **Question**: Write P in canonical form and calculate Q, R, and J. Recall the canonical form is

$$P = \begin{array}{c|c} Q & R \\ \hline 0 & J \end{array}$$

where J is an r by r identity, where r is the number of absorbing states.

Answer:

$$Q = \begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{6} \\ \frac{2}{6} \\ \frac{3}{6} \\ \frac{4}{6} \\ \frac{5}{6} \\ \frac{5}{6} \end{bmatrix}$$

$$J = [1]$$

[47]:
```
Q = P.copy()
Q.col_del(-1)
Q.row_del(-1)
Q
```

[47]:

$$\left[\begin{matrix}\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0\\\frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6}\\0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6}\\0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6}\\0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6}\\0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6}\\0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0\end{matrix}\right]$$

[25]:
```
R = P.col(-1)
R.row_del(-1)
R
```

[25]:

$$\left[\begin{matrix}0\\0\\\frac{1}{6}\\\frac{1}{3}\\\frac{1}{2}\\\frac{2}{3}\\\frac{5}{6}\\\frac{5}{6}\end{matrix}\right]$$

[27]:
```
J = P.row(-1).col(-1)
J
```

[27]:

$$\left[\begin{matrix}1\end{matrix}\right]$$

3. **Question**: Calculate N and B, where $N = (I - Q)^{-1}$ and $B = N \cdot R$.

[28]:
```
I8 = sympy.eye(8)
IminusQ = I8-Q
N = IminusQ.inv()
N
```

[28]:

$$\left[\begin{matrix}\frac{817473}{676630} & \frac{5517}{135326} & \frac{16551}{67663} & \frac{175587}{676630} & \frac{104232}{338315} & \frac{120108}{338315} & \frac{255141}{676630} & \frac{136227}{676630}\\\frac{68508}{338315} & \frac{68508}{67663} & \frac{5070}{67663} & \frac{83592}{338315} & \frac{90504}{338315} & \frac{106626}{338315} & \frac{119766}{338315} & \frac{108102}{338315}\\\frac{4593}{135326} & \frac{22965}{135326} & \frac{68895}{67663} & \frac{10191}{135326} & \frac{16794}{67663} & \frac{18090}{67663} & \frac{41541}{135326} & \frac{40119}{135326}\\\frac{3849}{676630} & \frac{3849}{135326} & \frac{11547}{67663} & \frac{688971}{676630} & \frac{25476}{338315} & \frac{83574}{338315} & \frac{175353}{676630} & \frac{173631}{676630}\\\frac{129}{135326} & \frac{645}{135326} & \frac{1935}{67663} & \frac{23091}{135326} & \frac{68886}{67663} & \frac{5016}{67663} & \frac{32457}{135326} & \frac{29235}{135326}\\\frac{54}{338315} & \frac{54}{67663} & \frac{324}{67663} & \frac{9666}{338315} & \frac{57672}{338315} & \frac{344088}{338315} & \frac{23028}{338315} & \frac{68886}{338315}\\\frac{9}{338315} & \frac{9}{67663} & \frac{54}{67663} & \frac{1611}{338315} & \frac{9612}{338315} & \frac{57348}{338315} & \frac{342153}{338315} & \frac{11481}{338315}\\\frac{3}{676630} & \frac{3}{135326} & \frac{9}{67663} & \frac{537}{676630} & \frac{1602}{338315} & \frac{9558}{338315} & \frac{114051}{676630} & \frac{680457}{676630}\end{matrix}\right]$$

[29]:
```
B = N*R
B
```

[29]:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

4. What can you interpret by looking at N?

- Specifically find the expected number of steps taken to reach a sum of 8 or more.

Answer: The expected number of steps taken to reach a sum of 8 or more, starting at sum 0 is given by the sum of the first row in N.

[30]:
```
ones8=sympy.ones(8,1)
expectedSteps = N * ones8
expectedSteps
```

[30]:

$$\begin{bmatrix} \frac{2026203}{676630} \\ \frac{944988}{338315} \\ \frac{326967}{135326} \\ \frac{1394619}{676630} \\ \frac{237231}{135326} \\ \frac{505284}{338315} \\ \frac{422529}{338315} \\ \frac{817473}{676630} \end{bmatrix}$$

Therefore the expected number of steps taken to reach 8 or more is 2.99

[33]:
```
2026203/676630
```

[33]: 2.99455093625763

0.2 Problem 2

Consider the random walk on [-3, -2, -1, 0, 1, 2, 3] as discussed in class. Specifically we start at any integer between -3 and 3 and we jump to its neighbors with equal probability, except when we reach -3, 3 where we keep repeating.

1. **Question**: Which states are absorbing states? Is the markov chain absorbing? If yes explain/prove why?

Answer: States -3,3 are absorbing states. To show that the markov chain is absorbing, we need to show that starting from any state we reach an absorbing state with non-zero probability. * If we start at -2, we can reach -3 with probability at least 0.5 * If we start at -1, we can reach -3 with probability at least 0.25 * If we start at 0, we can reach -3 with probability at least 0.125 * If we start at 1, we can reach 3 with probability at least 0.25 * If we start at 2, we can reach 3 with probability at least 0.5 * Since -3,3 are absorbing already, if we start at -3,3 we stay there with probability 1.

2. **Question**: Write the transition matrix P for this markov chain in canonical form. Recall the canonical form is

$$P = \begin{array}{c|c} Q & R \\ \hline 0 & J \end{array}$$

where J is an r by r identity, where r is the number of absorbing states. Explain your convention. Find Q, R, J. Preferably use Sympy or some other computer algebra software (say PARI/GP or SageMath) to find them as it will be helpful in the following question.

Answer: Say the rows are state -2, -1, 0, 1, 2, 3, -3 and the same for the columns

[51]:
```
half = sympy.Rational(1,2)
P = sympy.Matrix([
                  [0, half, 0, 0, 0, 0, half],
                  [ half, 0, half, 0, 0, 0, 0],
                  [ 0, half, 0, half, 0, 0, 0],
                  [0, 0, half, 0, half, 0, 0],
                  [0, 0, 0, half, 0, half, 0],
                  [0, 0, 0, 0, 0, 1, 0],
    [ 0, 0, 0, 0, 0, 0, 1]])
```

[60]:
```
P
```

[60]:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[61]:
```
Q = P.copy()
for i in range(2):
    Q.row_del(-1)
    Q.col_del(-1)
Q
```

[61]:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

[63]:
```
R = P.copy()
for i in range(2):
    R.row_del(-1)

for i in range(5):
    R.col_del(0)
R
```

[63]:

$$\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

[58]:

```
J = P.copy()
for i in range(5):
    J.row_del(0)
    J.col_del(0)
J
```

[58]:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. **Question** Compute N and B for this matrix P, where $N = (I - Q)^{-1}$ and $B = N \cdot R$.

[65]:

```
N = (sympy.eye(5)-Q).inv()
N
```

[65]:

$$\begin{bmatrix} \frac{5}{3} & \frac{4}{3} & 1 & \frac{2}{3} & \frac{1}{3} \\ \frac{4}{3} & \frac{8}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\ 1 & 2 & 3 & 2 & 1 \\ \frac{2}{3} & \frac{4}{3} & 2 & \frac{8}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \frac{5}{3} \end{bmatrix}$$

[68]:

```
B = N*R
B
```

[68]:

$$\begin{bmatrix} \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

4. **Question**:

- Suppose our markov chain starts at -2 and reaches the absorbing states. Find the probability of reaching the corresponding absorbing states. For example starting at -2 maybe one absorbing state has more chance than the other.
- Do the same for starting at 1.

Answer: * Starting at -2, the probability of getting absorbed at 3 is $\frac{1}{6}$ and getting absorbed at -3 is $\frac{5}{6}$. * Starting at 1, the probability of getting absorbed at 3 is $\frac{2}{3}$ and getting absorbed at -3 is $\frac{1}{3}$.

This is obtained by considering the entries of B.

5. **Question**

- Suppose our markov chain starts at 0. How many steps on average until it reaches an absorbing state?
- Find the same if the markov chain starts at 1.

Answer: We must sum the rows of Q.

[71]:
```
expectedSteps = N*sympy.ones(5,1)
expectedSteps
```

[71]:

$$\begin{bmatrix} 5 \\ 8 \\ 9 \\ 8 \\ 5 \end{bmatrix}$$

- Therefore we must take 9 steps on average until we reach an absorbing state starting from 0
- We must take 8 steps on average until we reach an absorbing state starting from 1.

[ ]:
```
```