% Graphs and Coloring % Junaid % Lecture 7: Discrete Mathematical Modelling

What is a Graph?

- A graph consists of two sets V and E where
- $$V = \{v_1, \ldots, v_n\}$$
- $$E = \{e_1, \ldots, e_m\}$$
- where each edge e_i consits of an **unordered** vertex pair $e_i = \{v_j, v_k\}$.
- $j \neq k$ because we avoid self loops.
- For a *directed* graph we consider edges as ordered pairs.

Example

- Let $V = \{1, 2, 3, 4, 5\}$ and
- $E = \{(1, 2), (1, 3), (2, 3), (2, 4)\}$.
- For an unordered graph, the order of the vertices in each edge does not matter.
- $G = \{V, E\}$ is a graph.

Drawing a graph

Let us draw

Complete Graphs

- A complete graph means that every vertex is connected to every other vertex.
- In other words for all i, j we have $e_k = \{v_i, v_j\} \in E$.
- We avoid self loops so $i \neq j$.
- We denote them by K_n.

Graph Vertex Coloring

- For a graph $G = \{V, E\}$ a *vertex coloring* is a choice of colors
- $$C = \{c_1, \ldots, c_p\}$$

- and an assignment of colors to vertices $f : V \to C$
- such that for any edge $e = \{v_i, v_j\}$,
- $$f(v_i) \neq f(v_j).$$
- In other words, for each edge we assign its constituent vertices with different colors.

Example

Question: Chromatic Number

- Given a graph $G = \{V, E\}$,
- what is the least number of colors to use to vertex color the graph?
- We denote the number by $\chi(G)$.
- It is called the *chromatic number* for the graph.

Bounds on Chromatic Number

- Claim: $\chi(G) \leq n$ where n is number of vertices of G.
- Proof:
- For complete graphs equality holds, $\chi(G) = n$.
- Can we find a better bound?

contd..

- Let us look at the degrees at each vertex.
- Here *degree* of a vertex is the number of edges emanating from the vertex.
- Example:

contd..

- Suppose $\Delta(G)$ is the largest degree of any vertex.
- $$\chi(G) \leq \Delta(G) + 1.$$
- Can we show this?
- Answer: Use a greedy algorithm.

Application: Class Scheduling

- If there are n classes.
- Students signup for them without knowing the time schedules.
- How many distinct time slots to pick?
- Answer:

Answer:

- Pick vertices as each class.
- Join two vertices by an edge if there is a student taking both classes.
- Chromatic number gives the minimum number of distinct time slots.

LP Formulation

- Can you formulate an LP for the above problem?
- Say we have a graph $G = (V, E)$.
- To start off we can pick n colors $\{c_1, \ldots, c_n\}$.
- Variables:
- $y[i] = 1$ if we use color i and 0 otherwise.
- $x[i, j] = 1$ if vertex i is assigned color j, 0 otherwise

Chromatic LP

- Minimize $\sum_i y[i]$.
- subject to
- $\sum_k x[ik] = 1$ for every i.
- $x[ik] \leq y[k]$
- $x[ik] + x[jk] \leq 1$ for $(v_i, v_k) \in E$ and each k.
- $y[k] \leq y[k-1]$.
- $x[ik], y[k] \in \{0, 1\}$.

Edge Coloring

- Question: How many colors do we need to color the edges of a graph such that no two edges of the same color connect at a vertex?
- Example:

Bounds on Edge Coloring

- We need at least $\Delta(G)$ colors.
- Infact this is a very good estimate.

- Vizing in 1964 proved that the number of colors needed is either
- $\Delta(G)$ or
- $\Delta(G) + 1$.

Edge Coloring Example

- Schedule matches in a tournament.
- Suppose we have 9 teams and we have a graph saying which teams must play each other.
- How many rounds of play needed to achieve this?

Example:

LP Formulation

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