Markov Chains: Expected Number of steps

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Math 381

Recap

- Last time we talked about Ergodicity and Left Eigenvectors.
- Let us ask a different question now?

How long?

- Suppose we have a Markov Chain with at least one absorbing state.
- If, starting in any state of the chain, it is possible to move to an absorbing state, then we say the chain is an **absorbing chain**.
- If we have an absorbing chain (like in the random walk on [-3, . . . , 3]), we may aks how long, on average, will it be until we are absorbed?

Example: Die Roll.

- Suppose we roll a die, and sum the results, until the sum is at least 8. How many rolls, on average will we make?
- Notice that it seems to be a very hard problem.
- There are so many possibilities like 6+2, 3+1+1+1+3, etc..
- We can model this with a Markov chain with nine states: 0, 1, 2, 3, 4, 5, 6, 7, and 8.
- The state indicated our current sum. Infact state 8 indicates the sum is at least 8.
- We start at state 0.
- State 8 is absorbing.

Die Roll Transition matrix

- Can you calculate the transition matrix ?
- $$P = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} & \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{3}{6} & \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} & \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \end{pmatrix}$$

Canonical Form of absorbing markov chain

- Last time we saw eigenvalues and eigenvectors.
- This time we will write P in a special form
- **Theorem:** Any transition matrix P of an absorbing markov chain can be expressed as

$$P = \begin{array}{c|c} Q & R \\ \hline 0 & J \end{array}$$

- Here I_r is the Identity matrix of r rows and columns where r is the number of absorbing states.
- Q, R are matrices with non-negative entries that arise from the transition probabilities between non-absorbing states.

Furthermore

- Furthermore, the series

$$N = I + Q + Q^2 + Q^3 + \cdots$$

  converges, and it equals
- $$N = (I - Q)^{-1}.$$
- The matrix N is important because

Theorem

- Let P be the transition matrix for an absorbing markov chain in canonical form.

$$P = \begin{array}{c|c} Q & R \\ \hline 0 & J \end{array}$$

- Let $N = (I - Q)^{-1}$. Then
- The ij-th entry of N is the expected number of times that the chain will be in state j after starting in state i.
- The sum of the i-th row of N gives the mean number of steps until absorbption when the chain is started in state i.
- The ij-th entry of the matrix $B = NR$ is the probability that, after starting in a non-absorbing state i, the process will end up in absorbing state j.

Application

- In the die example, the transition matrix P is
- $$P = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{3}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
- Can you find Q, R, J?

Contd..

- Let us find Q
- Q $=$

Contd..

- $N = (I - Q)^{-1}$
- Let us use sympy to compute N.

Contd..

- From the theorem, we know that we must sum the first row to get the expected number of steps until we are in an absorbing state.
- The first row of N sums to
- $\frac{776887}{279936} \approx 2.775.$
- Therefore on average, around 2.775 rolls until the sum is 8 or greater.

Another way

- Can you find P^8 in the case of die throw?
- Answer: It has first colums as all 0 and the last column is all 1.
- This is because after 8 throws we must be state 8.
- Since the game is over after a finite number of turns we can calculated expected number of throws until 8 in a direct way.
- We calculate the probability we reach state 8 after i throws.

contd..

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i	A	B
1	$A_{1,9}^{1} = 0$	$p_1 = 0$
2	$A_{1,9}^{2} = \frac{5}{12}$	$p_2 = A_{1,9}^{2} - A_{1,9}^{1} = \frac{5}{12}$
3	$A_{1,9}^{3} = \frac{181}{216}$	$p_3 = A_{1,9}^{3} - A_{1,9}^{2} = \frac{91}{216}$
4	$A_{1,9}^{4} = \frac{1261}{1296}$	$p_4 = A_{1,9}^{4} - A_{1,9}^{3} = \frac{175}{1296}$
5	$A_{1,9}^{5} = \frac{2585}{2592}$	$p_5 = A_{1,9}^{5} - A_{1,9}^{4} = \frac{7}{288}$
6	$A_{1,9}^{6} = \frac{46649}{46656}$	$p_6 = A_{1,9}^{6} - A_{1,9}^{5} = \frac{119}{46656}$
7	$A_{1,9}^{7} = \frac{279935}{279936}$	$p_7 = A_{1,9}^{7} - A_{1,9}^{6} = \frac{41}{279936}$
8	$A_{1,9}^{8} = 1$	$p_8 = A_{1,9}^{8} - A_{1,9}^{7} = \frac{1}{279936}$

- Column A gives the probability that we are in state 8 after i throws
- Column B gives the probability p_i that we are reach state 8 on throw i.

Expected number

- Then the expected number of throws until we reach 8 is \$$\sum_{i=0}^{8} i \cdot p_i = \frac{776887}{279936}$
- which agrees with the prior result.
- Note that the explicit computation works here because there are a finite number of steps until we reach 8, so we deal with a finite sum.
- For example the explicit method will not work when say you modify the problem and consider instead:
- If we throw 1 then we subtract from the sum (expect when sum is 0), if we throw 2,4,5,6 we add to the sum

Modified transition matrix

- Now the transition matrix is
- $\mathrm{B} =$

Contd..

- Note that now there is no upper bound on the number of rolls needed to reach 8 and hence the second method will not work.
- If you use the theorem you would get
- $\frac{2026203}{676630} \approx 2.99$.