Markov Chains : Ergodicity

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Math 381: Discrete Math Modelling

Markov Chains: Ergodicity

- Recall last time we discussed two kinds of states:
- **Recurrent** states.
- **Transient** states.
- In simple language a state s is **recurrent** if with probability 0 you ~~do not~~ escape s, once you start at s.
- Similarly a state s is **transient** if you can escape s (never return to s) after starting from s with a non-zero probability.
- Today we will attempt to prove a few observations from last time.
- Let us start with the notion of ergodicity.

Ergodicity

- Suppose P is a transition matrix for a markov chain M.
- Furthermore, let $P^n = (P_{ij}^{(n)})$ record the n-step transition probabilities.
- In other words, probability of going from state i to j after n-steps.
- Suppose P^n is **special**: the entries $P_{ij}^{(n)}$ are all non-zero.
- In other words, P allows movement from any state i to any state j after n steps.
- Then M is said to be **Ergodic**, if there exists an n such that P^n is special.

Example

- In simpler terms, M is **ergodic** if some power its transition matrix contains no zeros.
- Note that if P itself contains no zeros (for example the weather example from last time)

$$P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{pmatrix},$$

  then it is ergodic trivially.
- However, P may have zeros, but some power of P may still have all non-zero entries.
- Consider $P = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0 & 0.6 & 0.4 \\ 0.7 & 0 & 0.3 \end{pmatrix}$ then P has zero entries, but
- $P^2 = \begin{pmatrix} 0.04 & 0.64 & 0.32 \\ 0.28 & 0.36 & 0.36 \\ 0.35 & 0.56 & 0.09 \end{pmatrix}$ has non-zero entries, making P ergodic.

Meaning of Ergodicity

- Ergodicity is helpful because it gives a number n such that after n steps it is possible to go **from any state to any other state**.
- We saw the weather example from last time was ergodic for $n = 1$
- However, the random walk on $\{-3, -2, -1, 0, 1, 2, 3\}$ is **not ergodic**, because,
- for any power n there are zero entries.
- For example if n is even then cannot go from 1 to 2 (because 1 is odd, and 2 is even).
- Similarly, if n is odd then cannot go from 0 to 2 (because 0 and 2 are both even and n is odd).

Why Ergodicity?

- You may ask why do I need ergodicity.
- **Answer:**
- **Theorem:** For ergodic Markov chains we have a limiting distribution π, and every row of the matrix P^n converges to the limiting distribution.

Example

- Let us revisit the weather example.
- The matrix $P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$ had the property that for higher values of n
- $$P^n \approx P^{10} = \begin{pmatrix} 0.28395062 & 0.44444444 & 0.27160494 \\ 0.28395062 & 0.44444444 & 0.27160494 \\ 0.28395062 & 0.44444444 & 0.27160494 \end{pmatrix}$$
- In other words the limiting distribution in this case is $\pi = [0.28395062, 0.44444444, 0.27160494]$.
- This says that in the long run a typical day has 28% chance of being Sunny, 44% chance of being Cloudy and 27% chance of being Rainy.

Finding Limiting Distributions

- Suppose we are provided with a ergodic markov chain with transition matrix $P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$.
- And we must find the limit matrix

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{pmatrix}$$

- For example in the weather case we saw the answer is

$$P^n \approx P^{10} = \begin{pmatrix} 0.28395062 & 0.44444444 & 0.27160494 \\ 0.28395062 & 0.44444444 & 0.27160494 \\ 0.28395062 & 0.44444444 & 0.27160494 \end{pmatrix}$$

.

contd..

- Let $\overline{P} = \lim_{n\to\infty} P^n = \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{pmatrix}$ be the limit matrix.
- Then for any initial state v the final state is π

$$v\overline{P} = \pi.$$

- This means
  $\$\pi = v \lim_{n\to\infty} P^n = v \lim_{n+1\to\infty} P^{n+1} = v \lim_{n\to\infty} P^n P = v\overline{P}P = \pi P$
- In other words, we arrive at the surprising fact! that
- π is an **eigenvector** of P with **eigenvalue** 1.

Calculation

- Now let us solve for the eigenvector for $P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$.
- We want
$$\pi P = \pi$$
- In other words if $\pi = [\pi_1 \cdots \pi_k]$ then
$$\begin{bmatrix} \pi_1 & \cdots & \pi_k \end{bmatrix} \cdot \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \cdots & \pi_k \end{bmatrix}.$$
- In our case
$$\begin{aligned} 0.4\pi_1 + 0.2\pi_2 + 0.3\pi_3 &= \pi_1 \\ 0.4\pi_1 + 0.5\pi_2 + 0.4\pi_3 &= \pi_2 \\ 0.2\pi_1 + 0.3\pi_2 + 0.3\pi_3 &= \pi_3 \end{aligned}$$

contd..

- $$\begin{aligned} 0.4\pi_1 + 0.2\pi_2 + 0.3\pi_3 &= \pi_1 \\ 0.4\pi_1 + 0.5\pi_2 + 0.4\pi_3 &= \pi_2 \\ 0.2\pi_1 + 0.3\pi_2 + 0.3\pi_3 &= \pi_3 \end{aligned}$$
- After solving we get

$$\begin{aligned} \pi_1 &= \frac{23}{81} \quad \approx 0.28395 \\ \pi_2 &= \frac{36}{81} \quad \approx 0.44444 \\ \pi_3 &= \frac{22}{81} \quad \approx 0.2716 \end{aligned}$$

- which agrees with our initial guess!

Python Implementation

- One can use the `sympy` library to perform symbolic computations.
- We enter rational number for example $\frac{4}{10}$ by `sympy.Rational(4,10)`
- We create a matrix P by `P = sympy.Matrix()`
- Eigenvalues and eigenvectors can be computed by `P.eigenvals()` and `P.eigenvects()` respectively.
- Refer to jupyter notebook "Eigenvector Calculation" for full details.