MATH 224B
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Vector Fields: Practice Problems for Exam 2

Questions:

1. Indicate whether each of the following is a **scalar function,** or a **vector field** or a **nonsense** expression. Assume that $f(x, y, z)$ is a scalar function, $\mathbf{F}(x, y, z)$ and $\mathbf{G}(x, y, z)$ are vector fields. Assume they are nice, ie, continuous and have continuous partial derivatives of all orders.

   (a) $\operatorname{div}(\operatorname{grad} f)$ **Answer:** Scalar

   (b) $\operatorname{curl}(\nabla f)$ Vector

   (c) $\operatorname{curl}(\nabla \cdot f)$ Nonsense

   (d) $(\nabla \mathbf{F}) \cdot \mathbf{G}$ Nonsense

   (e) $(\nabla \cdot \mathbf{F})(\nabla \times \mathbf{G})$ Vector

   (f) $|\operatorname{curl}(\mathbf{F}) \times \mathbf{G}|$ Scalar

2. Among the expressions above which ones are always zero (either as scalar functions or as vector fields).

   Answer: Only b) is always 0

3. Let C be the curve consisting of a straight line segment from the origin to (2,0), then one quarter of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, -2)$.

   (a) Compute $\int_C x\,ds$. **Answer:** 6

   (b) Compute $\int_C x\,dy$. **Answer:** $-\pi$

4. Consider two vector fields $\mathbf{F} = \langle x + z, 1, x\rangle$ and $\mathbf{G} = \langle y, -x, e^z\rangle$ and $\mathbf{H} = \langle y, -x, \log z\rangle$.

   (a) For each of the aforementioned fields, determine whether it is conservative. Take the domain D of each to be all points in $\mathbb{R}^3$ where they are well-defined. If they are conservative, also find the potential. $\mathbf{F}$ is conservative, its potential $f = \frac{x^2}{2} + y + xz + c$, $\mathbf{G}$ and $\mathbf{H}$ are not conservative.

   (b) Let C be the curve from $(0, 0, 0)$ to $(4, 2, 20)$ along the intersection of the surfaces defined by $x^2 + y^2 = z$ and $x = 2y$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_C \mathbf{G} \cdot d\mathbf{r}$. **Answer:** 90 and $e^{20} - 1$

5. The function g of three variables is given by $g(x, y, z) = xz^2 + y - e^6$.

   (a) Suppose $\mathbf{r}(t)$ is a parameterized curve; we do not know the formula for $\mathbf{r}(t)$, but we know that $\mathbf{r}(5) = \langle 2, -7, 3\rangle$ and $\mathbf{r}'(5) = \langle -1, \pi, 2\rangle$. Define a new function $h(t) = g(\mathbf{r}(t))$; find $h'(5)$.

   Answer: $15 + \pi$

   (b) Find the equation of the tangent plane to the level set for g through the point $(2, -7, 3)$.

   Answer: $9x + y + 12z = 47$

   (c) Suppose you are at the point $(2, -7, 3)$ and you want to start moving in a direction so that g stays constant. Give one possible direction for which this is true. **Answer:** $\langle a, b, c\rangle$ such that $9a + b + 12c = 0$

6. Let S be the portion of the generalized cylinder $x^2 + 4z^2 = 16$ between $y = 0$ and $y = 10$, oriented by the outward normal.

   (a) Give a parameterization $\mathbf{r}(u, v)$ of S, including specifying the parameter space (ie, the domain of u, v). Does $\mathbf{r}_u \times \mathbf{r}_v$ give the orientation specified, or the opposite orientation? Explain briefly.

   Answer: $\mathbf{r} = \langle 4\cos v, u, 2\sin v\rangle$ $\quad 0 \le u \le 10, \quad 0 \le v \le 2\pi \quad$ $\mathbf{r}_u \times \mathbf{r}_v = \langle 2\cos v, 0, 4\sin v\rangle$
   Yes it does give us the required outward orientation

   (b) The boundary of S comes in two pieces C_1 in the $y = 0$ plane and C_2 in the $y = 10$ plane.

      i. Give a parameterization of C_1. Does your parameterization give the orientation of C_1 consistent with the given orientation of S or in the opposite orientation? Explain briefly.

      Answer: Opposite

      ii. Give a parameterization of C_2. Does your parameterization give the orientation of C_2 consistent with the given orientation of S, or the opposite orientation? Explain briefly. **Answer:** Same

7. Let S be the part of the surface $y = z^2$ inside the cylinder $x^2 + z^2 = 4$, oriented by the normal with positive $\mathbf{j}$ component.

   (a) Give a parameterization $\mathbf{r}(u, v)$ of S, including specifying the parameter space (ie, the domain of (u, v)). Does $\mathbf{r}_u \times \mathbf{r}_v$ give the orientation specified or the opposite orientation?

   Answer: $\mathbf{r}(u, v) = \langle u, v^2, v\rangle \quad u^2 + v^2 \le 4$ or
   in polar $\mathbf{r}(u, v) = \langle u\cos v, u^2\sin^2 v, u\sin v\rangle \quad 0 \le u \le 2, \quad 0 \le v \le 2\pi$

   (b) Give a parameterization of the boundary curve C of S as a function of t, including specifying the interval for t. Does your parameterization give the orientation of C consistent with the given orientation of S, or the opposite orientation?

   Answer: $\langle 2\cos t, 4\sin^2 t, 2\sin t\rangle \quad 0 \le t \le 2\pi$ and opposite orientation

   (c) Compute $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F} = \langle z, 4 - x^2 - z^2, -x\rangle$. (You may compute it directly or use one of the theorems of Stokes', Green's etc). **Answer:** 8π

8. Let S be the part of the cylinder $x^2 + y^2 = 9$, where $0 \le z \le 5$. Let $f(x, y, z) = 2z$, and let $\mathbf{F} = \langle 1, 0, 1\rangle$. Determine whether each of the following expressions make sense. If it does not make sense, say briefly why. If it does make sense, compute it. (Hint you may be able to reason directly from the meaning of surface integrals and compute them without setting up a parameterization.)

   (a) $\int\int_S f\, dS$ **Answer:** 150π

   (b) $\int\int_S f \cdot d\mathbf{S}$ **Answer:** Nonsense

   (c) $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ **Answer:** 0 because integral on one side will cancel with the other side.

9. Let S be the part of the surface $x = 1 - y^2 - z^2$ where $x \ge 0, z \ge 0$, and $0 \le y \le z$, oriented towards the origin. Compute $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x, y, z\rangle$. Be sure to explain your reasoning about orientation.

   Answer: $\frac{-3\pi}{16}$

10. Let C be the path that goes in a straight line from $(1, 0, 0)$ to $(0, -1, 0)$ to $(0, 0, 1)$ and back to $(1, 0, 0)$.

Use Stokes' Theorem to setup a double integral that computes

$$\int_C \langle xyz, x+y, x+z \rangle \cdot d\,\mathbf{r}\,.$$

Do not evaluate! Your answer should have two variables only and no vectors, looking something like $\int_{-}^{-}\int_{-}^{-} \text{—}\, dx\, dy$ **Answer:** $\int_0^1 \int_{x-1}^{0} (2 - 2xy + x^2 - x)\, dy\, dx$

11. Let C be the closed curve consisting of the line segments starting at the origin going to $(0,3,0)$, then to $(1,0,1)$, the back to the origin. Let $\mathbf{F} = \langle 2x - 3z, y + 7x, 5y - z \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\,\mathbf{r}$. You may compute it directly or use one of the theorems (Stokes', Green's, etc). It may also be helpful to note that C is a right triangle. **Answer:** -3