MATH 224B
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Multiple Integrals: Practice Problems for Exam 1

Questions:

1. In the following make a sketch of the region of integration and evaluate the double integrals:

   (a) $\iint_S (1+x)\sin y\, dx\, dy$, where S is the trapezoid with vertices $(0,0,(1,0),(1,2),(0,1)$.

   (b) $\iint_S e^{x+y}\, dx\, dy$, where $S = \{(x,y) \quad | \quad |x| + |y| \le 1\}$.

   Answer: a) $\frac{3}{2} + \cos 1 + \sin 1 - \cos 2 - 2\sin 2$. and b) $e - e^{-1}$.

2. In the following make a sketch of the region S and interchange the order of integration:

   (a) $\int_0^1 \left[\int_0^y f(x,y)\, dx\right] dy$.

   (b) $\int_1^e \left[\int_0^{\log x} f(x,y)\, dy\right] dx$.

   Answer: a) $\int_0^1 \int_x^1 f(x,y) dy dx$ and b) $\int_0^1 \int_{e^y}^e f(x,y) dx dy$.

3. When a double integral was set up for the volume V of the solid under the paraboloid $z = x^2 + y^2$ and above a region S of the xy-plane, the following sum of iterated integrals was obtained:

$$V = \int_0^1 \left[\int_0^y (x^2+y^2)\, dx\right] dy + \int_1^2 \left[\int_0^{2-y} (x^2+y^2)\, dx\right] dy.$$

   Sketch the region S and express V as an iterated integral in which the order of integration is reversed. Also carry out the integration and compute the volume V. **Answer:** $\int_0^1 \int_x^{2-x} x^2 + y^2 dy dx = \frac{4}{3}$.

4. When a double integral was set up for the volume V of the solid under the surface $z = f(x,y)$ and above a region S of the xy-plane, the following sum of iterated integrals was obtained:

$$V = \int_1^2 \left[\int_x^{x^3} f(x,y)\, dy\right] dx + \int_2^8 \left[\int_x^8 f(x,y)\, dy\right] dx.$$

   (a) Sketch the region S and express V as an iterated integral in which the order of integration is reversed.

   (b) Carry out the integration and compute V when $f(x,y) = \frac{6x(x^2+1)^2}{y}$.

   Answer: $\int_1^8 \int_{\sqrt[3]{y}}^{y} f(x,y)\, dx\, dy. = \frac{93555}{2}$.

5. Reverse the order of integration to derive the formula

$$\int_0^a \left[\int_0^y e^{m(a-x)} f(x)\, dx\right] dy = \int_0^a (a-x)e^{m(a-x)} f(x)\, dx.$$

6. Compute the following integral by changing to polar coordinates:

$$\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} dy\, dx.$$

Answer: $\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r\, dr\, d\theta = \frac{\pi a^2}{2}$.

7. Compute the volume of the region bounded by a circular cylinder $x^2+y^2=a^2$, the octant $x \geq 0, y \geq 0, z \geq 0$, and the plane $x+z=a$. **Answer:** $\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} \int_{z=0}^{a-x} dz\, r\, dr\, d\theta$.

8. Find the surface area of the portion of the sphere $x^2+y^2+z^2=a^2$ above the plane $z=\frac{a}{2}$. **Answer:** $\int_0^{2\pi} \int_0^{\frac{\sqrt{3}a}{2}} \frac{a}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$.

9. Interchange the order of integration to derive the formula

$$\int_0^x \left(\int_0^v \left[\int_0^u f(t)\, dt \right] du \right) dv = \frac{1}{2} \int_0^x (x-t)^2 f(t)\, dt.$$

10. Use a suitable linear transformation to evaluate the double integral

$$\iint_S (x-y)^2 \sin^2(x+y)\, dx\, dy,$$

where S is the parallelogram with vertices $(\pi, 0), (2\pi, \pi), (\pi, 2\pi), (0, \pi)$.

11. A solid is bounded by two concentric hemispheres of radii a and b, where $0 < a < b$. Find the center of mass if the density is constant. **Answer:** On the axis of symmetry, at a distance of $\frac{3}{8} \cdot \frac{b^4-a^4}{b^3-a^3}$.

12. The stem of a mushroom is a right circular cylinder of diameter 1 and length 2, and its cap is a hemisphere of radius R. If the mushroom is a homogenous solid with axial symmetry, and if its center of mass lies in the plane where the stem joins the cap, find R. **Answer:** $\sqrt[4]{2}$

13. Consider the mapping defined by the equations:

$$x = u+v, \quad y = v-u^2.$$

(a) Compute the Jacobian determinant $J = \frac{\partial(x,y)}{\partial(u,v)}$.

(b) A triangle T in the uv-plane has vertices $(0,0), (2,0), (0,2)$. Describe, by means of a sketch, its image S in the xy-plane.

(c) Calculate the area S by a double integral extended over S and also by a double integral extended over T.

(d) Evaluate $\iint_S (x-y+1)^{-2}\, dx\, dy$.

Answer: a) $1+2u$. and c) $\frac{14}{3}$. and d) $2 + \frac{2}{\sqrt{3}}\left(\arctan\frac{5}{\sqrt{3}} - \arctan\frac{1}{\sqrt{3}}\right)$.

14. Consider the mapping defined by the two equations $x=u^2-v^2, y=2uv$.

(a) Compute the Jacobian determinant $J = \frac{\partial(x,y)}{\partial(u,v)}$.

(b) Let T denote the rectangle in the uv-plane with vertices $(1,1),(2,1),(2,3),(1,3)$. Describe by means of a sketch, the image S in the xy-plane.

(c) Evaluate the double integral $\iint\limits_C xy\,dx\,dy$ by making the change of variables $x = u^2 - v^2, y = 2uv$, where $C = \{(x,y) \mid x^2 + y^2 \le 1\}$.

Answer: a) $4(u^2+v^2)$ and c) 0.

15. Find the center of mass of a thin plate in the shape of a rectangle $ABCD$ if the density at any point is the product of the distances of the point from the two adjacent sides AB and AD.

Answer: $\overline{x} = \frac{2}{3}|AB|$, $\overline{y} = \frac{2}{3}|AD|$, if AB and AD are along the x and y axes respectively.