Lecture - 9 Change of Variables Formula

Recall 'u' substitution

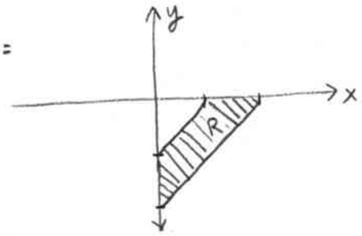
<u>Suppose</u> $$\int_0^1 \sqrt{x^2+1}\cdot x\,dx = \int_1^2 \frac{1}{2}\sqrt{u}\,du$$

This means that integrating the function $(\sqrt{x^2+1})\cdot x$ from 0 to 1 is the same as integrating the simpler function $\frac{1}{2}\sqrt{u}$ from 1 to 2

We will explore this idea in $\iint$ and $\iiint$ today.

<u>Question - 1</u>.

Suppose we ask the following question: Evaluate $\iint_R e^{\frac{x+y}{x-y}}\,dA$ where

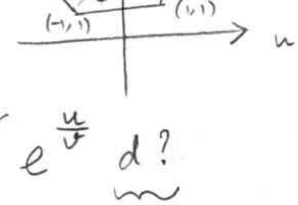
$R =$  the trapezoidal region given by $(1,0), (2,0), (0,-2), (0,-1)$.

How do we proceed?

The function looks like $e^{\frac{x+y}{x-y}}$ and there does not seem an easy way to integrate this. As we shall see in the lecture today, the solution goes via a step like u substitution.

We set $u = x+y$
$v = x-y$

and the region (x,y) in 

corresponds to the region 

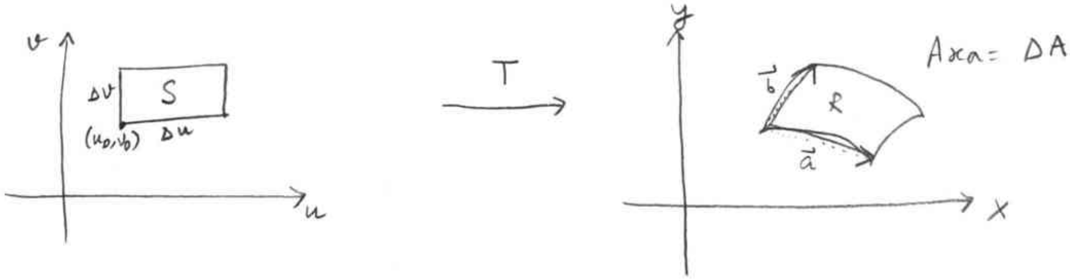
therefore $$\iint_R e^{\frac{x+y}{x-y}}\,dx\,dy = \iint_S e^{\frac{u}{v}}\,d\,?$$

what to do here? $du\,dv$ merely will not work!

(Think about how $u = x^2+1$ gives $du = 2x\,dx$, so, $du \leftrightarrows 2x\,dx$ not just dx)

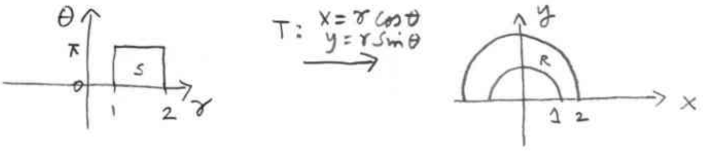
Change of Variables : a closer look

Suppose



T is an invertible (one-one, onto) function that maps the region S in (u,v) coordinates to the region R in (x,y) Coordinates.

[As an example recall how



]

Let $\quad x = g(u,v)$

$\qquad\; y = h(u,v)$

The bottom edge $\xrightarrow[\Delta u]{} \hat{\imath}$ of the rectangle S is mapped to $\vec{a}$

The left edge $\Delta v \uparrow \hat{\jmath}$ of the rectangle S is mapped to $\vec{b}$.

Can we find $\vec{a}$ and $\vec{b}$?

<u>Homework:</u> Try to estimate $\vec{a}$ and $\vec{b}$?

You should see that $\vec{a}$ is approximately $\left(\dfrac{\Delta g}{\Delta u}\hat{\imath} + \dfrac{\Delta h}{\Delta u}\hat{\jmath}\right)\Delta u$

and $\vec{b}$ is approximately $\left(\dfrac{\Delta g}{\Delta v}\hat{\imath} + \dfrac{\Delta h}{\Delta v}\hat{\jmath}.\right)\Delta v$

This means that the area of R is approximately the determinant of

$$|\vec{a}\times\vec{b}| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\Delta g}{\Delta u} & \frac{\Delta h}{\Delta u} & 0 \\ \frac{\Delta g}{\Delta v} & \frac{\Delta h}{\Delta v} & 0 \end{vmatrix} \Delta u \Delta v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\Delta x}{\Delta u} & \frac{\Delta y}{\Delta u} & 0 \\ \frac{\Delta x}{\Delta v} & \frac{\Delta y}{\Delta v} & 0 \end{vmatrix} \Delta u \Delta v$$

In the limit this becomes

$$\Delta A = \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} \right| \Delta u \Delta v$$

$$= \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k} \right| \Delta u \Delta v$$

$(\det A = \det A^T)$

$$= \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \Delta u \Delta v$$

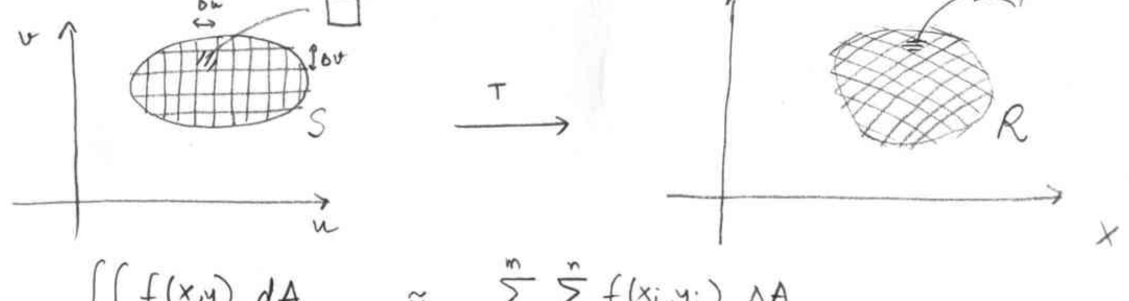
__Definition__ The Jacobian of the transformation $T: \begin{array}{l} x = g(u,v) \\ y = h(u,v) \end{array}$ is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation

$$\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

__As a consequence__



$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{j=1}^{n} \sum_{i=1}^{n} f(g(u_i,v_j), h(u_i,v_j)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

$$= \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Therefore we obtain the following :

Theorem: Change of variables in the double integral : If T is a transformation with continuous partial derivatives (nice transformation) whose Jacobian is non-zero and T maps a region S in (u,v) plane to R on (x,y) plane.

Then
$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Solution of Question 1

Now, we can complete Question - 1

$$\iint_R e^{\frac{x+y}{x-y}}\, dx\, dy = \iint_S e^{\frac{u}{v}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

What is $\dfrac{\partial(x,y)}{\partial(u,v)}$?

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$u = x + y$

$v = x - y$

$u + v = 2x$, $2y = u - v$

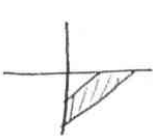
$x = \dfrac{u+v}{2}$, $y = \dfrac{u-v}{2}$

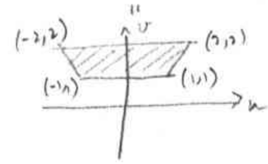
$$\Rightarrow \frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial v} = \frac{1}{2}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2} \quad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

Therefore

$$\iint_R e^{\frac{x+y}{x-y}}\, dx\, dy = \iint_S e^{\frac{u}{v}} \left(\frac{1}{2}\right) du\, dv$$





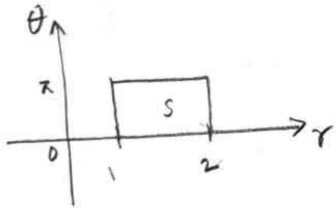
$$= \int_{v=1}^{2} \int_{u=-v}^{u=v} e^{u/v}\ \frac{1}{2}\ du\, dv$$

$$= \int_1^2 \frac{1}{2}\ v e^{u/v} \Big|_{u=-v}^{u=v} dv$$

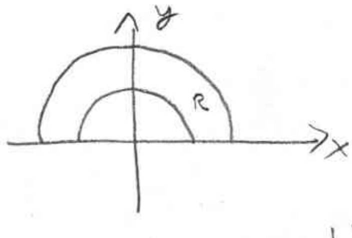
$$= \frac{1}{2} \int_1^2 (v.e^1 - v e^{-1})\, dv$$

$$= \frac{1}{2}\left(e - \frac{1}{e}\right) \int_1^2 v\, dv = \frac{3}{4}\left(e - \frac{1}{e}\right)$$

Cartesian to Polar



$$\xrightarrow[T]{x = r\cos\theta,\ y = r\sin\theta}$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

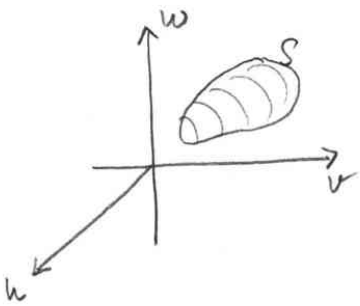
$$= r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

Therefore

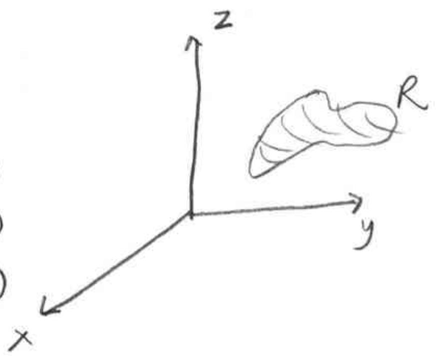
$$\iint_R f(x,y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta)\ r.dr\, d\theta$$

Triple integrals



$$\xrightarrow{T}$$

$$x = g(u,v,w)$$
$$y = h(u,v,w)$$
$$z = k(u,v,w)$$



The Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Therefore

$$\iiint_R f(x,y,z)\, dV = \iiint_S f\big(x(u,v,w),\ y(u,v,w),\ z(u,v,w)\big) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Example Spherical to Cartesian

$$x = r \sin\varphi \cos\theta$$
$$y = r \sin\varphi \sin\theta$$
$$z = r \cos\varphi$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & -r\sin\theta\sin\varphi & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\cos\theta\sin\varphi & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix}$$

(Simplify)

$$= -r^2 \sin\varphi\cos^2\varphi - r^2\sin\varphi\sin^2\varphi = -r^2\sin\varphi$$

$$\therefore \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = |-r^2\sin\varphi| = r^2\sin\varphi.$$

$$\therefore \iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(r,\theta,\varphi)\, r^2\sin\varphi\, dr\,d\theta\,d$$