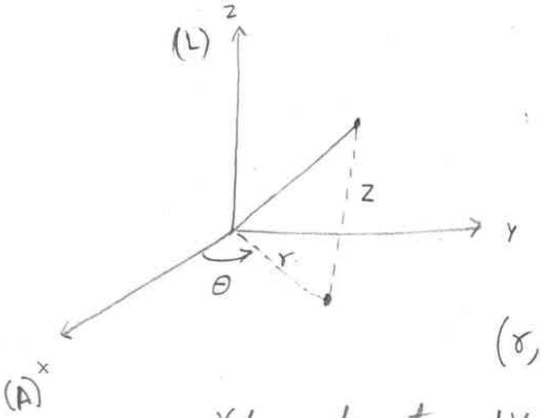
Lecture-8

Triple Integrals in Cylindrical and Spherical coordinates

Cylindrical Coordinates



$$r = \sqrt{x^2+y^2}$$

$$\tan\theta = \frac{y}{x}$$

$$z = z$$

X axis : polar axis A

Z axis : longitudinal axis L

$$x = r\cos\theta$$

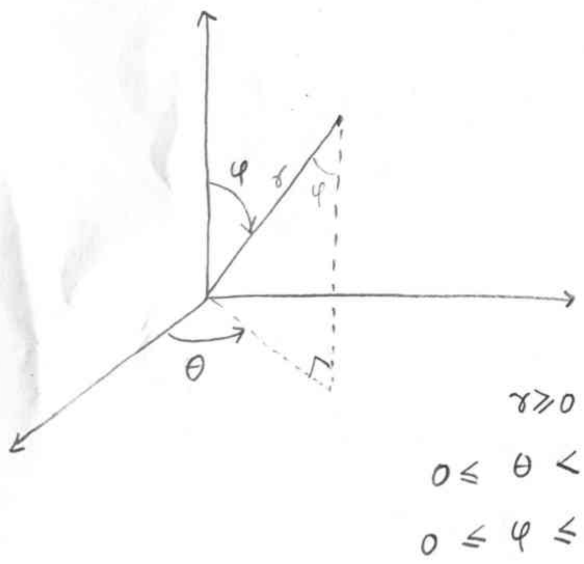
$$y = r\sin\theta$$

$$z = z$$

(r, θ, z) is the Cylindrical Coordinate for (x, y, z)

Volume element $dV = r\,dr\,d\theta\,dz$ (Proof next page)

Spherical Coordinates



r: radial distance

θ: azimuthal angle (angle of rotation from initial meridian plane (xz plane))

φ: polar angle (angle w.r.t. polar axis (z axis))

$$r \geq 0$$

$$0 \leq \theta < 2\pi$$

$$0 \leq \varphi \leq \pi$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\tan\theta = \frac{y}{x}$$

$$\tan\varphi = \frac{\sqrt{x^2+y^2}}{z} \quad \text{or,}$$

$$\cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}} \quad \text{or,}$$

$$\sin\varphi = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$x = r\sin\varphi\cos\theta$$

$$y = r\sin\varphi\sin\theta$$

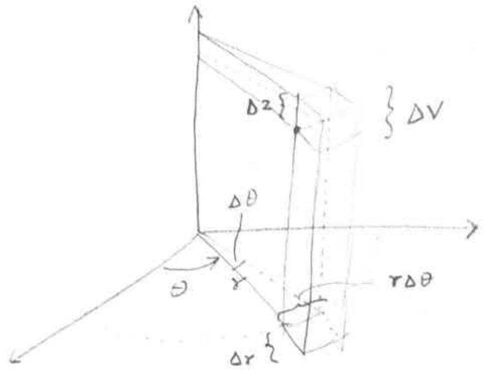
$$z = r\cos\varphi.$$

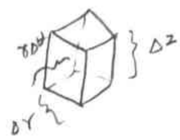
Volume element $dV = r^2\sin\varphi\,dr\,d\theta\,d\varphi$.

(Proof next page)

Volume element ΔV

- In cylindrical coordinates





The cuboid has sides Δr, $r\Delta\theta$, Δz

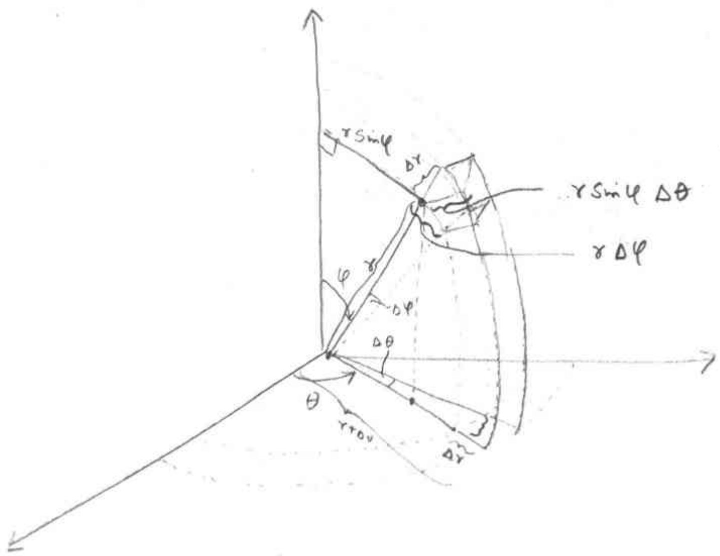
Therefore Volume $\Delta V = \Delta r \cdot r\Delta\theta \cdot \Delta z$

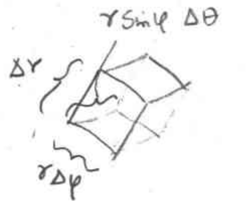
In the limit therefore

$$dV = r\, dr\, d\theta\, dz$$

$$I = \iiint_V F(\theta, r, z)\, r\, d\theta\, dr\, dz$$

- In spherical coordinates





$$\Delta V = \Delta r \cdot r\Delta\varphi \cdot r\sin\varphi\,\Delta\theta$$

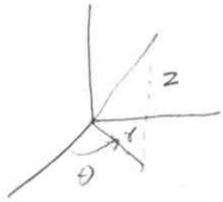
In the limit therefore

$$dV = r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

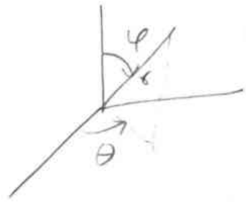
$$I = \iiint_V F(\theta, r, \varphi)\, r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

Plot the points

cylindrical coordinate $(2, 2\pi/3, 1)$
(r, θ, z)



Spherical coordinate $(2, \pi/4, \pi/3)$
(r, θ, φ)



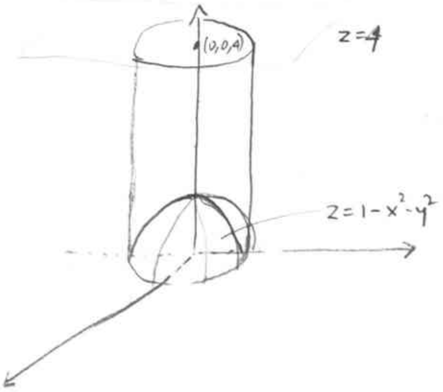
Find spherical coordinats of $(x,y,z) = (0, 2\sqrt{3}, -2)$

· Exercise : Find center of mass of a solid hemisphere radius a . using spherical coordinates.

Find center of mass of a solid hemisphere with density proportional to distance from base using spherical coordinates.

Examples

A solid E lies within the cylinder $x^2+y^2=1$, below plane $z=4$, above paraboloid $z=1-x^2-y^2$. The density at any point is proportional to the distance from the axis of the cylinder. Find the mass.



In cylindrical coordinate

- cylinder : $r=1$
- plane $z=4$
- paraboloid $z=1-r^2$

$$\therefore \quad E=\{(r,\theta,z) \mid 0\le\theta\le 2\pi,\ 0\le r\le 1,\ 1-r^2\le z\le 4\}$$

density $\rho(x,y,z)=K\sqrt{x^2+y^2}$

$$\therefore \quad \rho(r,\theta,z)=K\cdot r$$

$$\therefore \quad \text{mass}=\iiint_E Kr\, dV=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=1-r^2}^{4} Kr\; r\,dz\,dr\,d\theta$$

$$= \cdots$$

(complete)

$$=$$

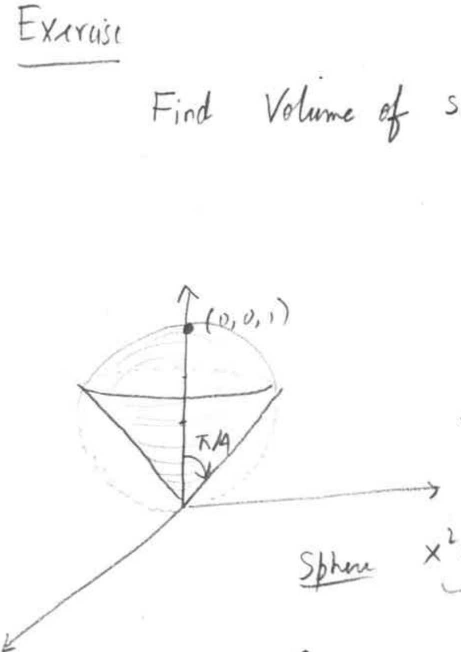
$$=\frac{12\pi K}{5}$$

Exercise

Find Volume of solid that lies above the cone $z = \sqrt{x^2+y^2}$
below the sphere $\underbrace{x^2+y^2+z^2 = z}_{\parallel}$

$$x^2+y^2+z^2-2\cdot z\cdot\tfrac{1}{2}+\tfrac{1}{4}=\tfrac{1}{4}$$

$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2=\tfrac{1}{4}$$



Sphere $\underbrace{x^2+y^2}_{}+z^2 = z$

$r^2 = r\cos\varphi \Rightarrow r = \cos\varphi$

Cone: $Z = \sqrt{x^2+y^2}$

$$\underbrace{r\cos\varphi}_{z} = \underbrace{\sqrt{r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta}}_{\sqrt{x^2+y^2}} = r\sin\varphi$$

$$\Rightarrow \cos\varphi = \sin\varphi \Rightarrow \varphi = \pi/4$$

$$E = \left\{ (r,\theta,\varphi) \;\middle|\; 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi/4,\ 0 \le r \le \cos\varphi \right\}$$

$$\therefore V(E) = \iiint_E dV = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/4}\int_{r=0}^{\cos\varphi} r^2\sin\varphi\, dr\, d\varphi\, d\theta$$

$$= \text{(compute ...)}$$

$$= \pi/8$$