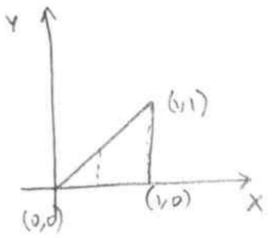
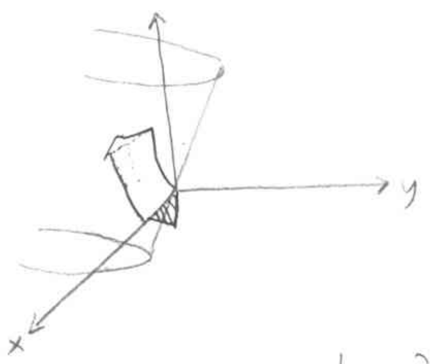
Lecture 6

Surface Area (Problems)

- Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the XY plane with vertices $(0,0), (1,0), (1,1)$





$$\text{Surface area} = \iint_T \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2$$

$$= \int_{x=0}^{1} \int_{y=0}^{y=x} \sqrt{1 + 4x^2 + 4}\ dy\ dx$$

$$= \int_0^1 \left(\sqrt{5 + 4x^2}\right) x\ dx$$

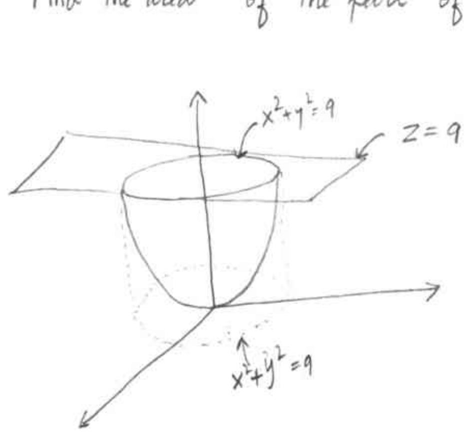
$$5 + 4x^2 = u \qquad 8x\,dx = du$$

$$= \int_5^9 \frac{1}{8} \sqrt{u}\ du$$

$$= \frac{1}{4\,\cancel{8}} \frac{u^{3/2}}{\frac{3}{\cancel{2}}} \Bigg|_5^9$$

$$= \frac{27}{4} - \frac{5\sqrt{5}}{4}$$

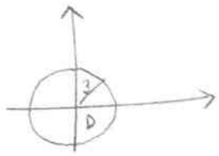
• Find the area of the part of the paraboloid $z=x^2+y^2$ under the plane $z=9$



$z=x^2+y^2$

$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial y} = 2y$



$$\iint_D \sqrt{1+4x^2+4y^2}\ dA$$

$$= \int_0^{2\pi}\int_0^{3} \sqrt{1+4r^2}\ r\,dr\ d\theta$$

$1+4r^2=u$

$8r\,dr = du$

$$= \int_0^{2\pi}\int_1^{37} \sqrt{u}\ \frac{1}{8}du\ d\theta$$

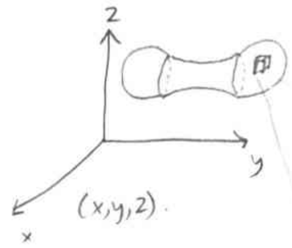
$$= \frac{1}{\cancel{8}_{\cancel{4}_2}} \frac{37^{3/2}-1}{\frac{3}{\cancel{2}}} \cdot \cancel{2}\pi$$

$$= \left(37\sqrt{37}-1\right)\frac{\pi}{6}.$$

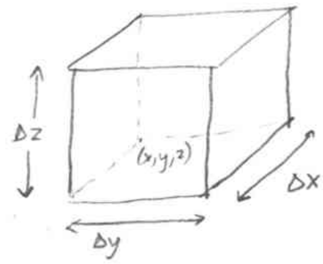
• Exercise. Let $z=ax+by+c$ be a plane. Show that the area on the plane $z=ax+by+c$ above any region D in the xy-plane is $\sqrt{1+a^2+b^2}\cdot A(D)$ where $A(D)$ is the area of D.

Introduction to triple integrals

Suppose we have a solid E shaped



and let $\rho(x,y,z)$ be its density at a point (x,y,z). Suppose we want to compute the total mass of the solid. We can draw a small cube



Since the cube is small, we can assume $\rho(x,y,z)$ does not change much in the cube. Therefore mass of the cube

$$\Delta m = \underbrace{\rho(x,y,z)}_{\text{density.}} \; \underbrace{\Delta x \Delta y \Delta z}_{\text{Volume of cube.}}$$

(Since density of solid $= \frac{\text{mass}}{\text{volume}}$)

$$\text{Therefore } M = \iiint_E \rho(x,y,z)\, dV =$$

$$= \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \rho(x_{ijk}, y_{ijk}, z_{ijk}) \cdot \Delta V$$

Remark

Provided ρ is constant and E is bounded, we can integrate with respect to any variable first. In fact there are 6 different ways to order the integral

- $dx\,dy\,dz$
- $dx\,dz\,dy$
- $dy\,dx\,dz$
- $dy\,dz\,dx$
- $dz\,dx\,dy$
- $dz\,dy\,dx$

(Fubini for triple integrals!)

• **Example** Suppose we want to find the mass of a box filled with some material of density e^z. The box is located

$$B = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 3,\ 0 \le z \le 2\}$$

$$\text{mass} = \iiint_B e^z \, dV$$

$$= \int_0^2 \int_{-1}^3 \int_0^1 e^z \, dx\, dy\, dz$$

$$= \int_0^2 \int_{-1}^3 e^z [x]\Big|_0^1 \, dy\, dz$$

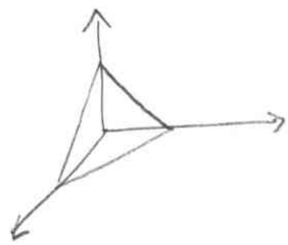
$$= \int_0^2 e^z \cdot 1 \cdot y\Big|_{-1}^3 \, dz$$

$$= \int_0^2 4e^z \, dz$$

$$= 4 \, e^z\Big|_0^2 = 4(e^2 - 1)$$

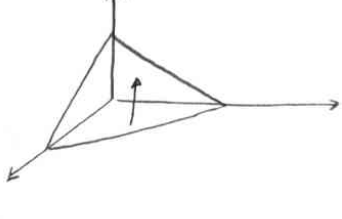
Example

• Find the $\iiint_E z \, dV$ where E is the solid tetrahedron bounded by $x=0, y=0, z=0$ and $x+y+z=1$



STEP1 • DRAW A PICTURE

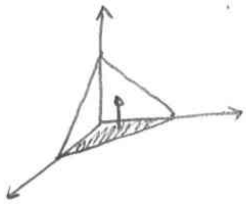
STEP2 • Pick a direction to integrate, let integrate w.r.t. z first

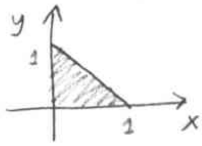


z starts at 0

ends at $x+y+z=1 \Rightarrow z = 1-x-y$

· Question: For any (x,y) in the shadow of E on the xy plane does z goes from 0 to $1-x-y$



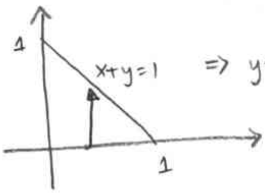


Ans: Yes!

Therefore

$$\iiint_E z \, dV = \iint \underbrace{\int_{z=0}^{z=1-x-y} z \, dz}_{\checkmark} \; d? \; d?$$

STEP3. Pick the next direction to integrate. Lets by y. Notice we are looking at the "shadow".



For any x between 0 and 1, y goes from 0 to $1-x$.

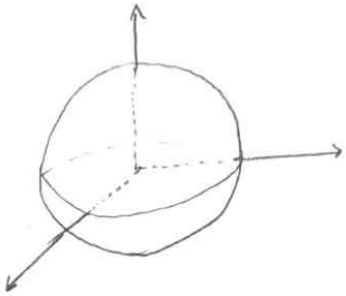
Therefore the integral

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{z=1-x-y} z \, dz \; dy \; dx$$

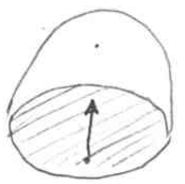
Compute !

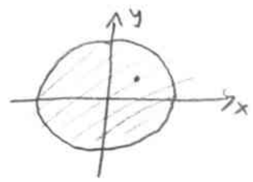
Example

- Find the Volume of the sphere of radius R.



Let us compute volume of hemisphere E. Note that computing volume means $\iiint_E dV$ (integrand is 1).





Let us integrate z first

For any (x,y) z goes from 0 to $z=\sqrt{R^2-x^2-y^2}$

Therefore

Volume of hemisphere

$$\int\int\int_{z=0}^{\sqrt{R^2-x^2-y^2}} dz \quad d?\ d?$$

Now on the shadow

instead of integrating in cartesian, lets integrate in polar $\begin{matrix} 0 \le r \le R \\ 0 \le \theta \le 2\pi \end{matrix}$ and $x^2+y^2=r^2$ $\Rightarrow \sqrt{R^2-x^2-y^2}=\sqrt{R^2-r^2}$

$$\int_0^{2\pi}\int_0^{R}\int_{z=0}^{\sqrt{R^2-r^2}} dz \quad r\,dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{R} \sqrt{R^2-r^2} \quad r\,dr\, d\theta.$$

HW: Complete.