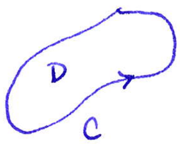
Divergence Theorem

Recall Green's theorem



$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

The left hand side is actually

$$\oint_C \vec{F}\cdot d\vec{r} \qquad \text{where} \qquad \vec{F} = P\hat{\imath} + Q\hat{\jmath}.$$

The right hand side is actually

$(\text{curl } \vec{F})\cdot \vec{k}$ where $\vec{k}$ is the unit vector perpendicular to the xy plane (the plane containing the region)

In other words, we obtain a vector form:

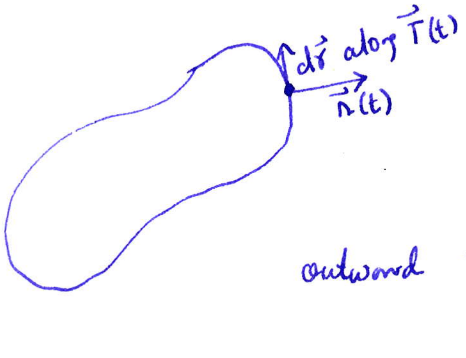
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (\text{curl } \vec{F})\cdot \hat{k}\, dA$$

This is generalized to the stokes' theorem which we saw last time.

Let us now investigate a slightly different integral.

Instead of looking at $\vec{F}\cdot d\vec{r}$ (the tangential component of $\vec{F}$)

suppose we were interested in the normal component.

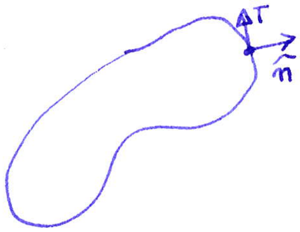


ie, $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$

unit tangent $\vec{T}(t) = \dfrac{x'(t)\,\hat{\imath}}{|\vec{r}'(t)|} + \dfrac{y'(t)}{|\vec{r}'(t)|}\hat{\jmath}$

outward normal vector $\hat{n} = \dfrac{y'(t)}{|\vec{r}'(t)|}\hat{\imath} - \dfrac{x'(t)}{|\vec{r}'(t)|}\hat{\jmath}$

Therefore,



$$\oint \vec{F}\cdot\hat{n}\, ds = \int_{t=a}^{t=b} (\vec{F}\cdot\hat{n})\, |\vec{r}\,'(t)|\, dt$$

$$= \int_a^b \left(P\frac{dy}{dt} - Q\frac{dx}{dt}\right)\frac{1}{|\vec{r}\,'(t)|}\, |\vec{r}\,'(t)|\, dt$$

$$= \oint_C P\,dy - Q\,dx$$

$$\underset{\text{By green's theorem}}{=} \iint_D \frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right)\, dA$$

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

But $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is just $\operatorname{div} \vec{F}$

$$\therefore \oint_C \vec{F}\cdot\hat{n}\, ds = \iint_D \operatorname{div} \vec{F}(x,y)\, dA$$

In words: line integral of the normal component of $\vec{F}$ along C is equal to the ~~dt~~ double integral of the divergence of $\vec{F}$ over the region D enclosed by C.

The divergence theorem is an extension of the idea to 3D. It says:

If E is a simple solid region and S be its boundary surface of E with positive orientation (outward). Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region containg E ($\vec{F}$ is nice)

Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\ dV$$

Thus the divergence theorem states that the flux of $\vec{F}$ across a boundaey surface of E is equal to the 6ifle integral of the divergence of $\vec{F}$ over E.

<u>Ex</u> Flux of $\vec{F} = \langle z, y, x\rangle$ over unit sphere $x^2+y^2+z^2=1$

<u>using $\iint_S \vec{F}\cdot d\vec{S}$</u>

$$x = \sin\varphi\cos\theta \qquad \vec{r}_\varphi = \langle \cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi\rangle$$
$$y = \sin\varphi\sin\theta \qquad \vec{r}_\theta = \langle -\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0\rangle$$
$$z = \cos\varphi$$

$$d\vec{S} = \vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix} = \langle +\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi\rangle$$

$$(\text{or, } \vec{r}_\theta \times \vec{r}_\varphi)$$

$$\therefore \iint \vec{F}\cdot d\vec{S} = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} \langle\cos\varphi,\ \sin\varphi\sin\theta,\ \sin\varphi\cos\theta\rangle\cdot\langle\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi\rangle \ \cancel{\text{outward}}\ d\varphi\, d\theta$$

$$= \int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi} \sin^2\varphi\cos\varphi\underbrace{\cos\theta\, d\theta}_{0}\, d\varphi + \int_0^{\pi}\int_0^{2\pi} \sin^3\varphi\sin^2\theta\, d\theta\, d\varphi + \int_0^{\pi}\int_0^{2\pi} \sin^2\varphi\cos\varphi\underbrace{\cos\theta\, d\theta}_{0}\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} \sin^3\varphi\sin^2\theta\, d\theta\, d\varphi = \frac{4}{3}\cdot\pi = \frac{4\pi}{3}$$

~~Using~~ The RHS

$$\iiint \operatorname{div}\vec{F}\, dV$$

$\vec{F} = \langle z, y, x \rangle \qquad \Rightarrow \operatorname{div}\vec{F} = \frac{\partial}{\partial x} z + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} x$

$$= 0 + 1 + 0$$

$$= 1$$

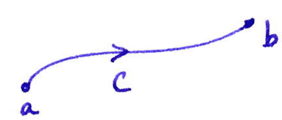
$\therefore \quad \iiint \operatorname{div}\vec{F}\, dV = \iiint 1\, d\vec{V} =$ Volume of sphere

$$= \frac{4\pi}{3}.$$

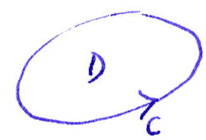
Summary

- **FTC** $\quad \displaystyle\int_a^b F'(x)\, dx = F(b) - F(a)$

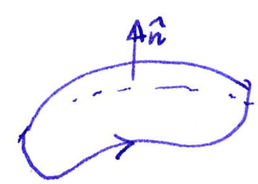
- **FT Line Integrals** $\quad \displaystyle\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$



- **Green's Theorem** $\quad \displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$



- **Stokes' Theorem** $\quad \displaystyle\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$



- **Divergence Theorem**

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}$$