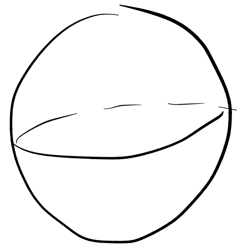
Lecture-21 - Surface integrals of vector fields

Recap the notion of orientability: An orientable surface is one that has a well defined outside vs inside or outword vs downward.

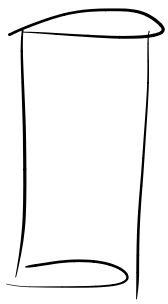
Example of orientable surfaces

Most nice surfaces:

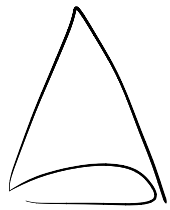
Sphere



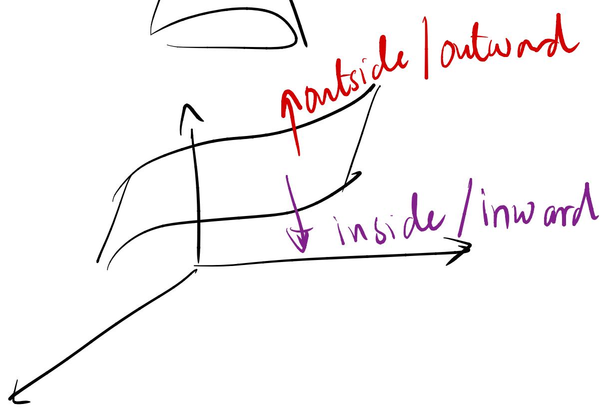
Cyllinder



Cone

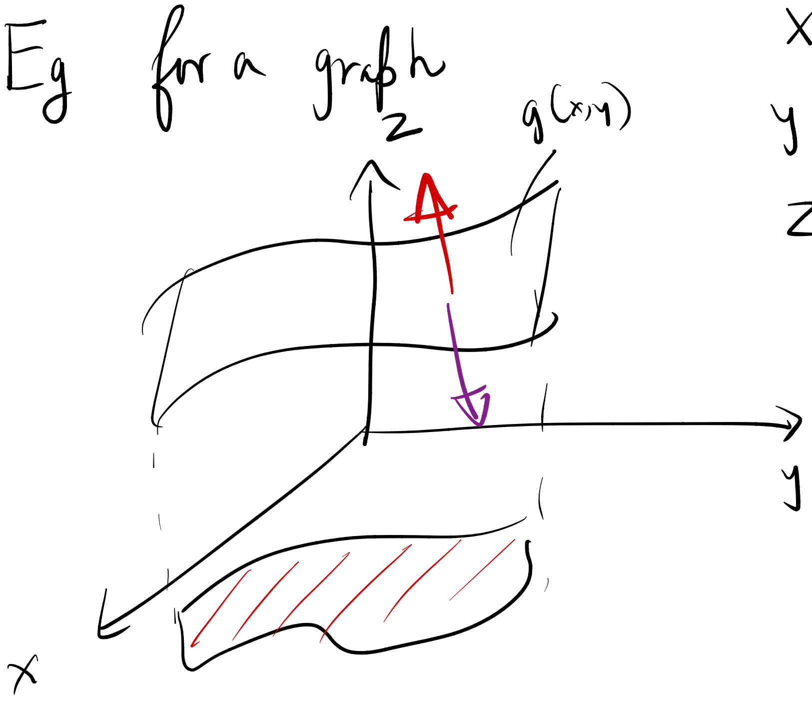


Graph of a function



The normal vector (may be outside or inside depending on parametrization) is a crucial tool in detecting orientation.

Eg for a graph

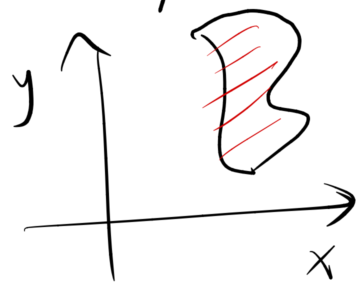


$x = x$

$y = y$

$z = g(x,y)$

here the parameter space is



and a normal vector is

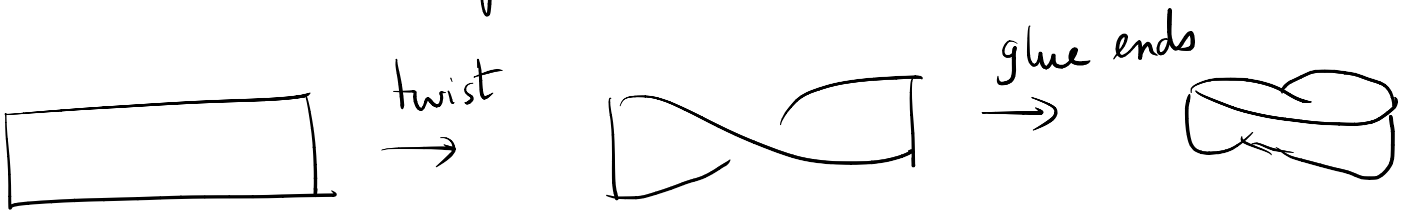
$$\vec{r}_x \times \vec{r}_y = \langle 1, 0, g_x \rangle \times \langle 0, 1, g_y \rangle$$

$$= \langle -g_x, -g_y, 1 \rangle \uparrow$$

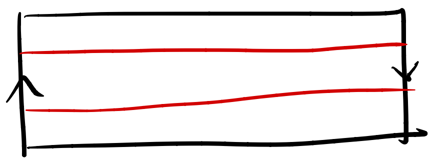
The z coordinate of $\underline{1}$ indicates outward

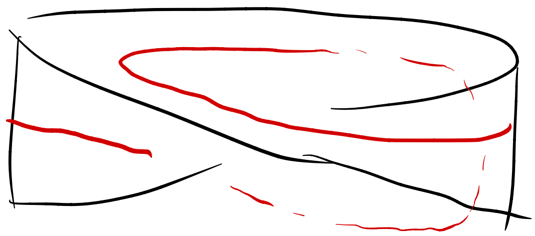
$\therefore \vec{r}_y \times \vec{r}_x = \langle g_x, g_y, -1 \rangle$ will be inward $\downarrow$

A non-orientable surface is a Möbius strip



This surface has no well defined outside vs inside because on travelling on this surface on can start at an outside and then suddenly be inside due to the twist.

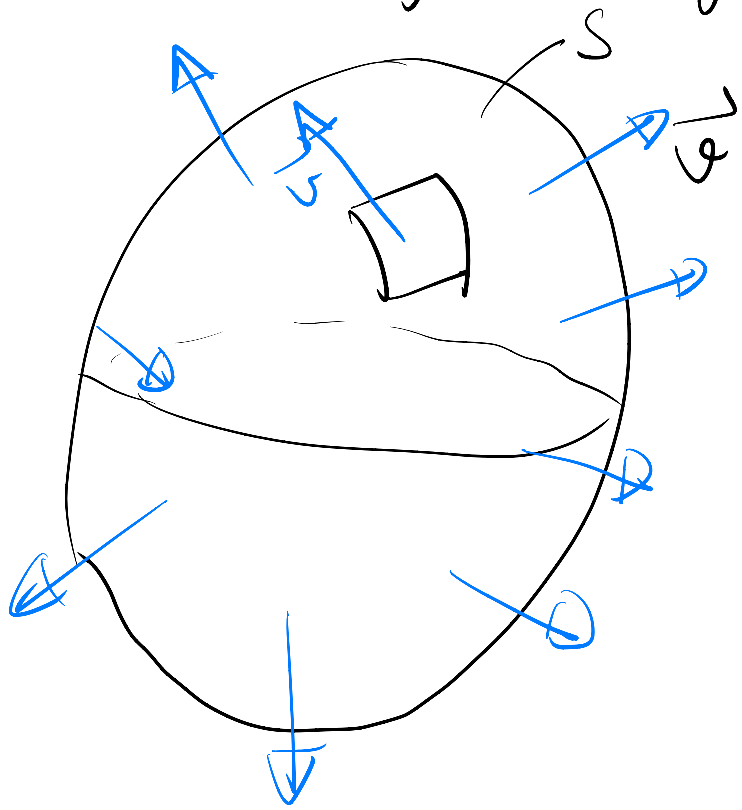




We will be working over nice ie, orientable surfaces and this anomaly will not arise.

Flux ie, Surface integral of vector field

Suppose we have a vector field $\vec{v}$ that models fluid velocity and say we pick a surface S and want to study amount of water escaping the surface



If the density (mass per unit volume) of the fluid is $\rho(x,y,z)$, then in unit time mass of fluid escaping perpendicular to the area is

$$\Delta m = \rho(x,y,z)\left(\vec{v} \cdot \hat{n}\right) \Delta S$$

The dot product is needed because we want to measure the amount of fluid moving out (ie, perpendicular) to ΔS,

In the limit

$$m = \int_S \underbrace{\rho(x,y,z)\,\vec{v}}_{\vec{F}} \cdot \underbrace{\hat{n}\, dS}_{\vec{dS}}$$

Therefoe we arrive at the following definition

If $\vec{F}$ is a vector field defined on an oriented surface S with unit normal $\hat{n}$ then the surface integral of $\vec{F}$ over S (flux of $\vec{F}$ across S) is given by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS$$

In most cases we can compute $\hat{n}\, dS$ via

$$\vec{r}_u \times \vec{r}_v = \hat{n}\, dS$$

or,

$$\vec{r}_v \times \vec{r}_u = \hat{n}\, dS$$

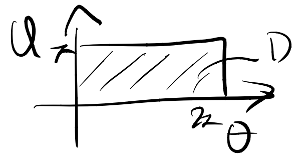
depending on the orientation outside or inside.

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, dA$$

Find Flux of vector field $\vec{F}(x,y,z) = z\hat{i} + y\hat{j} + x\hat{k}$

across $x^2+y^2+z^2=1$

- Parametrically $r(\theta,\varphi) = \langle \sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi \rangle$



$\Leftarrow$ $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$

- $\vec{F}(\vec{r}(\theta,\varphi)) = \cos\varphi\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \sin\varphi\cos\theta\,\hat{k}$

- $\vec{r}_\varphi \times \vec{r}_\theta = \langle \sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi \rangle$

(Note that $\vec{r}_\theta \times \vec{r}_\varphi$ would have all components with negative sign and would point inward)

$\therefore \quad \vec{F}(\vec{r}(\theta,\varphi)) \cdot (\vec{r}_\varphi \times \vec{r}_\theta)$

$$= \cos\varphi \sin^2\varphi \cos\theta + \sin\varphi \sin\theta \sin^2\varphi \sin\theta$$
$$+ \sin^2\varphi \cos\theta \cos\varphi$$
$$= 2\sin^2\varphi \cos\theta \cos\varphi + \sin^3\varphi \sin^2\theta$$

Therefore

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_\varphi \times \vec{r}_\theta)\, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} \left(2\sin^2\varphi \cos\varphi \cos\theta + \sin^3\varphi \sin^2\theta\right) d\varphi\, d\theta$$

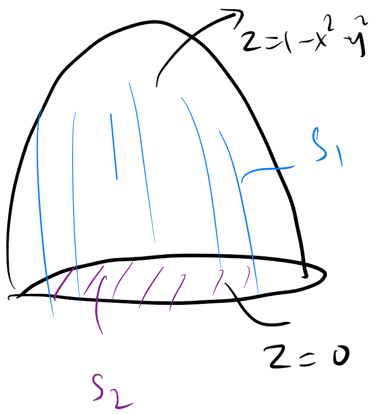
$$= \int_0^{2\pi} 2\cos\theta \int 2\sin^2\varphi \cos\varphi\, d\varphi\, d\theta + \int_0^{2\pi} \sin^2\theta \int_0^{\pi} \sin^3\varphi\, d\varphi\, d\theta$$

$$= 0 + \pi \cdot \frac{4}{3}$$

Often times we may have to split the surface into cases.

Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = y\hat{i} + x\hat{j} + z\hat{k}$

and S is the boundary of the solid enclosed by paraboloid $z = 1 - x^2 - y^2$ and plane $z = 0$



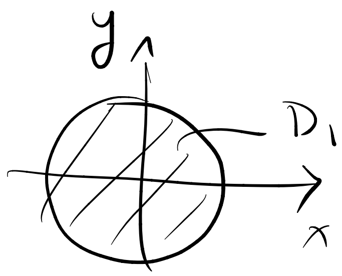
The flux will be

$$\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$$

For S_1

$$\vec{r}(x,y) = \langle x, y, 1 - x^2 - y^2 \rangle$$

Parameter space



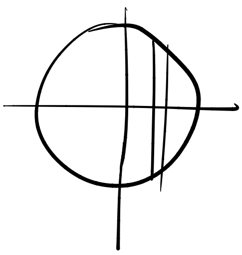
$$\vec{r}_x = \langle 1, 0, -2x \rangle, \quad \vec{r}_y = \langle 0, 1, -2y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1 \rangle$$

$$\vec{F}(\vec{r}(x,y)) = \langle y, x, 1-x^2-y^2 \rangle$$

$$\therefore \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{D_1} \vec{F}(\vec{r}(x,y))\cdot(\vec{r}_x\times\vec{r}_y)\,dA$$

$$= \iint_{D_1} \left(2xy + 2xy + 1 - x^2 - y^2\right) dA$$



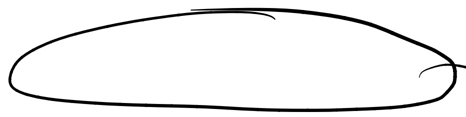
$$= \int_{-1}^{1}\int_{x=\sqrt{1-x^2}}^{x=\sqrt{1-x^2}} \left(1 + 4xy - x^2 - y^2\right) dy\,dx$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left(1 + 4r^2\cos\theta\sin\theta - r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} (r - r^3) + 4r^2\sin\theta\cos\theta\,dr\,d\theta$$

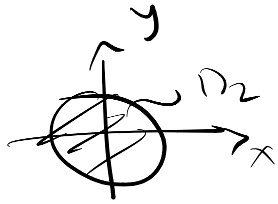
$$= \frac{1}{4}\int_0^{2\pi} d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

For S_2



$z=0$

$\vec{r}(x,y) = \langle x, y, 0 \rangle$



$\vec{r}_x = \langle 1, 0, 0 \rangle$

$\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle$

$\vec{r}_y = \langle 0, 1, 0 \rangle$

Outward will be

$r_y \times r_x = \langle 0, 0, -1 \rangle$

$$\therefore \iint_{D_2} \vec{F}(r(x,y)) \cdot \vec{r}_x \times \vec{r}_y \, dA$$

$$= \iint_{D_2} \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dA$$

$$= \iint_{D_2} 0 \, dA = 0$$

$$\therefore \text{Total flux} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$