Lecture -16- Green's Theorem

Let us recall that a vector field $\vec{F}$ is conservative if it has a potential function f ie, $\vec{F} = \vec{\nabla} f$

For a 2D vector field $\vec{F} = P\hat{i} + Q\hat{j}$

this amounts to checking if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

How to construct potential function

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{j}$$

and $\vec{\nabla} f = \dfrac{\partial f}{\partial x}\hat{i} + \dfrac{\partial f}{\partial y}\hat{j} + \dfrac{\partial f}{\partial z}\hat{k}$

Therefore, $\dfrac{\partial f}{\partial x} = P$, $\dfrac{\partial f}{\partial y} = Q$, $\dfrac{\partial f}{\partial z} = R$

gives us the potential function by integrating.

Example Find potential function for

$$\vec{F} = \underbrace{(2x\cos(y) - 2z^3)}_{\frac{\partial f}{\partial x}}\hat{i} + \underbrace{(3 + 2ye^z - x^2\sin y)}_{\frac{\partial f}{\partial y}}\hat{j} + \underbrace{(y^2e^z - 6xz^2)}_{\frac{\partial f}{\partial z}}\hat{k}$$

Ans: $\dfrac{\partial f}{\partial x} = 2x\cos y - 2z^3$

① $\Rightarrow$ $f(x,y,z) = 2\cos y \cdot \dfrac{x^2}{2} - 2z^3x + h(y,z)$

$\dfrac{\partial f}{\partial y} = 3 + 2ye^z - x^2\sin y$

② $\Rightarrow$ $f(x,y,z) = 3y + \dfrac{2e^z y^2}{2} + x^2\cos y + k(x,z)$

$\dfrac{\partial f}{\partial z} = y^2e^z - 6xz^2$

③ $\Rightarrow$ $f(x,y,z) = y^2e^z - \dfrac{6xz^3}{3} + \ell(x,y)$

Comparing ① and ② $h(y,z) = 3y + e^z y^2 + c$

$k(x,z) = -2z^3x + c$

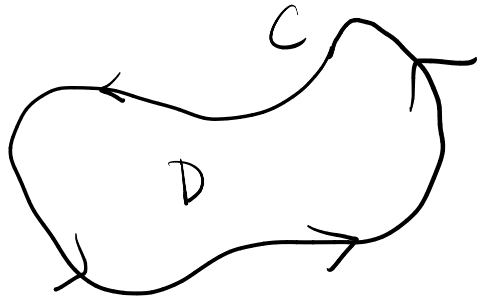
② and ③ $k(x,z) = -2z^3x + c$

$\ell(x,y) = 3y + x^2\cos y + c$

Therefore

$$f(x,y,z) = y^2 e^z - 2xz^3 + 3y + x^2\cos y + C$$

Greens theorem



It converts line integrals to area integrals.

If C is a positively oriented, piecewise smooth, simple, closed curve and D be the region enclosed by the curve, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Alternative notation :

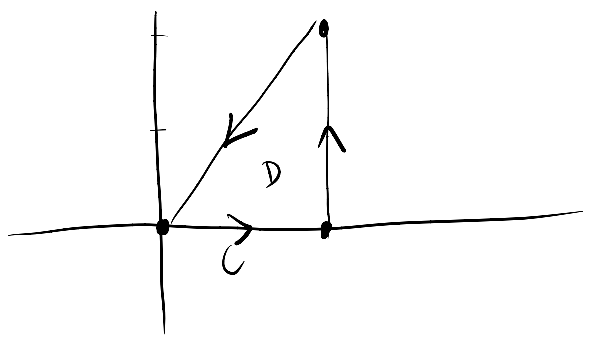
$$\oint_C P\,dx + Q\,dy \quad \text{or,} \quad \oint_C P\,dx + Q\,dy$$

<u>Example</u>: Use Green's theorem to evaluate

$$\oint_C xy\,dx + x^2y^3\,dy$$ where C is the triangle with

vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation

<u>Ans</u>

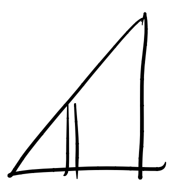


Region on the left $\Rightarrow$ positive orientation.

$P = xy$ $\qquad \dfrac{\partial P}{\partial y} = x$, $\qquad \dfrac{\partial Q}{\partial x} = 2xy^3$

$Q = x^2y^3$

$$\oint_C xy\,dx + x^2y^3\,dy = \iint_D (2xy^3 - x)\,dA$$



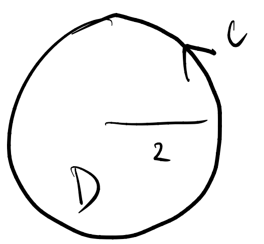
$$= \int_{x=0}^{1}\int_{y=0}^{2x} (2xy^3 - x)\,dy\,dx$$

$$= \int_0^1 \left[\cancel{2}\frac{xy^4}{\cancel{4}\,2} - xy\right]\Bigg|_0^{2x} dx$$

$$= \int_0^1 \left(8x^5 - 2x^2\right) dx = \frac{2}{3}$$

Example 2: Evaluate $\oint_C y^3\,dx - x^3\,dy$ where C is positively oriented circle of radius 2 centered at origin

Ans



$$P = y^3 \qquad \frac{\partial P}{\partial y} = 3y^2$$

$$Q = -x^3 \qquad \frac{\partial Q}{\partial x} = -3x^2$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint \left(-3x^2 - 3y^2\right) dA$$

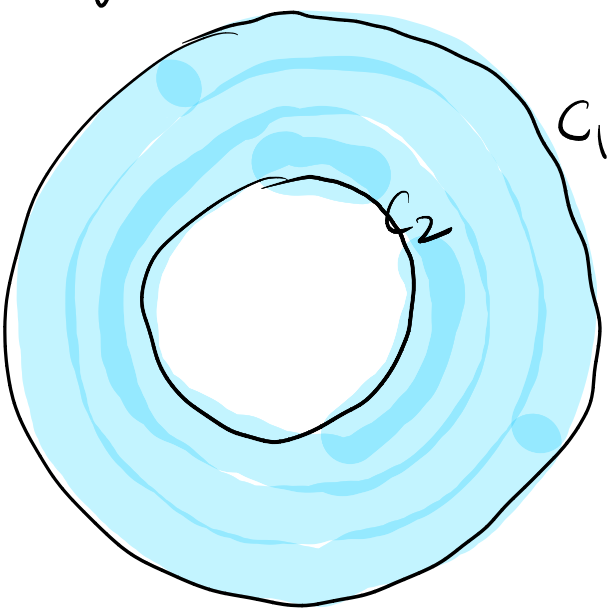
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} -3r^2 \; r\,dr\,d\theta$$

$$= -\int_0^{2\pi} \left(3 \cdot \frac{r^4}{4} \Big|_0^2 \right) d\theta$$

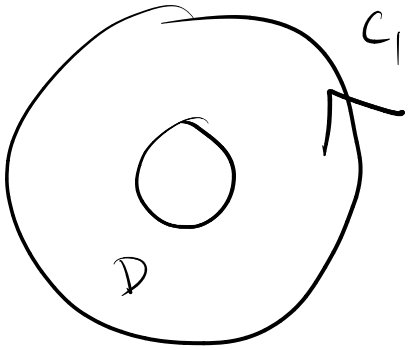
$$= -12 \cdot 2\pi = -24\pi$$

Green theorem for more general regions (with holes)

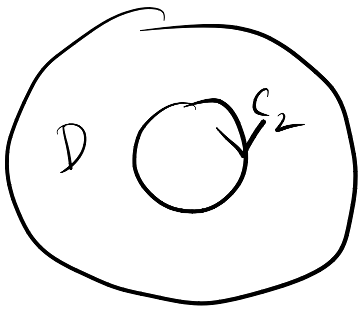
Suppose



Positive orientation on C_1 is



Positive orientation on C_2 is

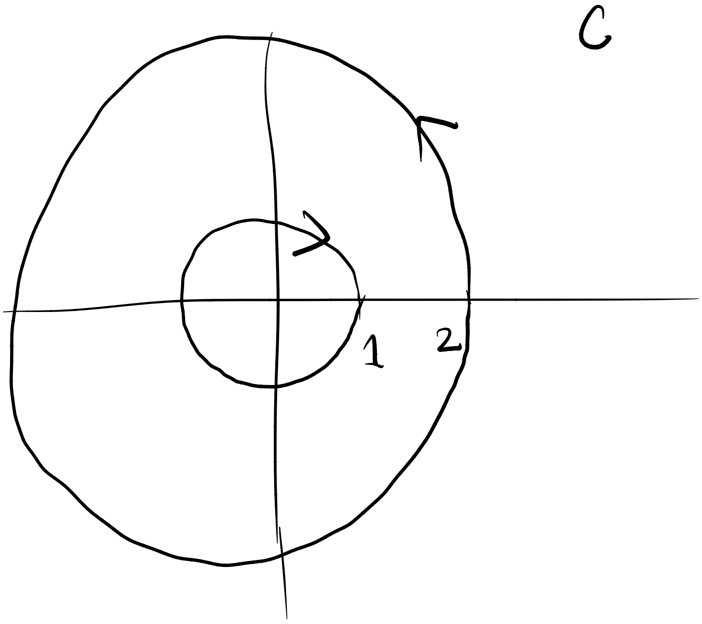


Because the Blue region should be on the left.

The green's theorem is

$$\oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Example Evaluate $\oint_C y^3\,dx - x^3\,dy$ for C being the two circles of radius 1 and 2 with positive orientation.



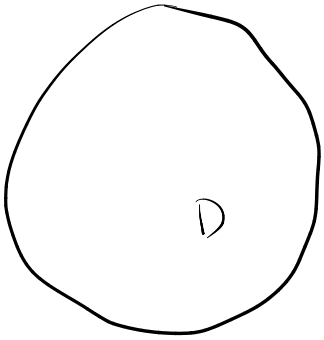
$$\oint_C y^3\,dx - x^3\,dy = -3\iint_D (x^2+y^2)\,dA$$

$$= -3\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta$$

$$= -3\int_0^{2\pi} \frac{1}{4}r^4\Big|_1^2\,d\theta$$

$$= -\frac{45}{2}\pi$$

Example: Use Green's theorem to find the area of disk of radius a



$$\iint_D 1 \cdot dA = ?$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

we can take $Q = x$ and $P = 0$

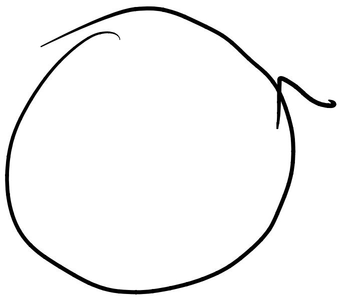
or, $Q = 0$, $P = -y$

or, $Q = \frac{x}{2}$ $P = -\frac{y}{2}$ and so on

$\therefore$ If we take $Q = x$, $P = 0$ we have

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C x \, dy$$

$$= ?$$



$$\int_0^{2\pi} a\cos t\,(a\cos t\,dt)$$

$x = a\cos t$

$y = a\sin t$

$0 \leq t \leq 2\pi$

$$= a^2 \int_0^{2\pi} \cos^2 t \, dt$$

$$= a^2 \left(\frac{\pi}{4}\right) \cdot 4 = \pi a^2$$

<u>Example</u> Find area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Try this