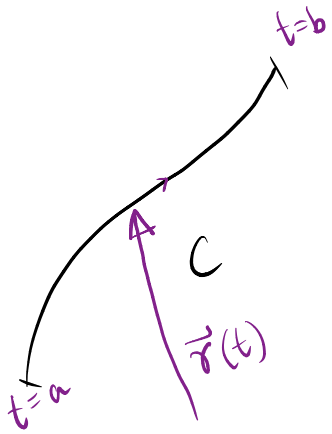
Lecture - 15 - Fundamental Theorem For Line Integrals

<u>Recall</u> In Calculus we know,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

similar holds for <u>Line integrals of vector fields</u>



If C is a smooth curve given by $\vec{r}(t)$; $a \le t \le b$ and f is a function whose gradient $\vec{\nabla} f$ is continuous on C, then

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

<u>The proof goes like</u>:

$$\int_C \vec{\nabla} f \cdot d\vec{r} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) dt$$

$$= \int_a^b \left(\frac{\partial f}{\cancel{\partial x}} \cdot \frac{\cancel{dx}}{dt} + \frac{\partial f}{\cancel{\partial y}} \cdot \frac{\cancel{dy}}{dt} + \frac{\partial f}{\cancel{\partial z}} \cdot \frac{\cancel{dz}}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} \left[f(\vec{r}(t)) \right] dt$$

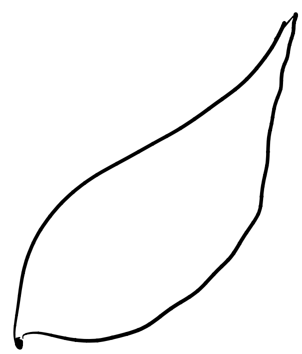
$$= f(\vec{r}(b)) - f(\vec{r}(a))$$ by Fundamental theorem of Calculus.

<u>Example</u> : Evaluate $\int_C \vec{\nabla} f \cdot d\vec{r}$ where

$f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is <u>any path</u> that starts at $(1, \frac{1}{2}, 2)$ ends at $(2, 1, -1)$

<u>Solution</u>

$(2,1,-1)$



$(1, \frac{1}{2}, 2)$

Let C be any path from $(1, \frac{1}{2}, 2)$ to $(2,1,-1)$, $\vec{r}(t)$ be the path for $a \le t \le b$.

ie, $\vec{r}(a) = (1, \frac{1}{2}, 2)$

$\vec{r}(b) = (2, 1, -1)$

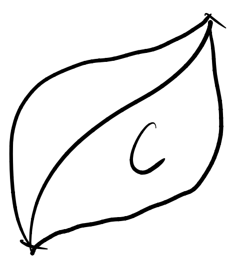
Then $$\int_C \vec{\nabla} f \cdot d\vec{r} = f(2,1,-1) - f(1, \tfrac{1}{2}, 2)$$

$$= \cos(2\pi) + \sin\pi - 2(1)(-1) - \left(\cos\pi + \sin\tfrac{\pi}{2} - 1\left(\tfrac{1}{2}\right)\right) = 4$$

The key takeaway is that for these line integrals the path is not important.

Conservative vector fields

Conservative vector fields are those vector fields $\vec{F}$ whose $\int_C \vec{F} \cdot d\vec{r}$ does not depend on the path.



From the previous discussion, we can see why the following is true,

If $\vec{F}$ is conservative vector field then $\vec{F} = \vec{\nabla} f$

($\vec{F}$ is the gradient of a potential function)

and then it is clear why $\int \vec{F} \cdot d\vec{r}$ does not depend on path since $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r}$ which does not depend on path.

How to check if vector field is conservative?

- For 2D vector fields $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ the check is easy:

$$\text{If } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ then}$$

vector field is conservative.

Caution! This works on a region that is simply connected (no holes!) (See Lecture 19)

Lets do some examples : Determine if $\vec{F}$ is conservative on $\mathbb{R}^2$

a) $\vec{F}(x,y) = (x^2 - yx)\hat{\imath} + (y^2 - xy)\hat{\jmath}$

b) $\vec{F}(x,y) = \left(2xe^{xy} + x^2ye^{xy}\right)\hat{\imath} + \left(x^3e^{xy} + 2y\right)\hat{\jmath}$

Try doing this.

Ans a) is not conservative

b) is conservative.

How to find the potential function ie, f such that $\vec{\nabla} f = \vec{F}$.

<u>Ans:</u> For 2D: $\vec{F} = P\hat{i} + Q\hat{j}$ its easy:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Therefore $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$f(x,y) = \int P(x,y)\,dx \ ; \quad f(x,y) = \int Q(x,y)\,dy$$

<u>Example</u>