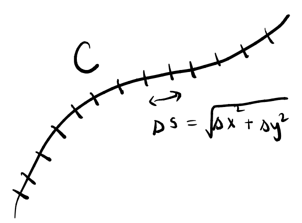
Recap

Line integral w.r.t. arc length



$$\int_C f(x,y)\, ds = \int_C f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Line integral w.r.t. x and y

$$\int_C P\,dx + Q\,dy = \int_C P\,dx + \int_C Q\,dy$$

Example a) Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$, where C is the line segment from (0,2) to (1,4).

b) Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$, where C is the line segment from (1,4) to (0,2)

a) $\vec{r}(t) = (1-t)\langle 0,2\rangle + t\langle 1,4\rangle, \quad 0 \le t \le 1$

$\Rightarrow \quad x(t) = (1-t)\,0 + t\cdot 1 = t \quad \Rightarrow \frac{dx}{dt} = 1$

$\quad\quad y(t) = (1-t)\,2 + t\cdot 4 = 2+2t \quad \Rightarrow \frac{dy}{dt} = 2$

$$\int_C \sin(\pi y)\,dy + \int_C yx^2\,dx = \int_{t=0}^{t=1} \sin(\pi(2+2t))\,\frac{dy}{dt}\cdot dt + \int_{t=0}^{t=1} (2+2t)(t^2)\,\frac{dx}{dt}\,dt$$

$$= \int_0^1 \sin 2\pi \underbrace{(1+t)}_{\substack{1+t=u \\ du = dt, \\ 1 \le u \le 2}} \cdot 2 \cdot dt \quad + \int_0^1 2t^2 (1+t) \cdot 1 \cdot dt$$

$$= \int_{u=1}^{u=2} 2 \sin 2\pi u \; du \quad + \int_0^1 2t^2 \, dt \quad + \int_0^1 2t^3 \, dt$$

$$= -2 \frac{\cos 2\pi u}{2\pi} \Big|_1^2 \quad + 2\frac{t^3}{3} \Big|_0^1 \quad + 2\frac{t^4}{4} \Big|_0^1$$

$$= -\frac{1}{\pi}\left[1 - 1\right] \quad + \frac{2}{3} + \frac{2}{4} \quad = \frac{14}{12} = \frac{7}{6}$$

b) $\vec{r}(t) = (1-t)\langle 1, 4\rangle + t\langle 0, 2\rangle$, $\quad 0 \le t \le 1$

$x(t) = (1-t)(1) + t \cdot 0 = 1-t \qquad \frac{dx}{dt} = -1$

$y(t) = (1-t)(4) + t \cdot 2 = 4-2t \qquad \frac{dy}{dt} = -2$

$$\int_C \sin \pi y \, dy + \int_C y x^2 dx = \int_{t=0}^1 \sin \pi (4-2t) \frac{dy}{dt} \cdot dt$$

$$+ \int_{t=0}^1 (4-2t)(1-t)^2 \frac{dx}{dt} \cdot dt$$

$$= \int_0^1 \sin 2\pi \underbrace{(2-t)}_{u} (-2) \, dt \quad + \int_0^1 2(2-t) \underbrace{(1-t)}_{u}{}^2 (-1) \, dt$$

$du = -2\pi \, dt$

$\Rightarrow dt = -\frac{du}{2\pi}$

$dt = -du$

$t=0 \Rightarrow u=1$

$t=1 \Rightarrow u=0$

$$\int_{4\pi}^{2\pi} \sin u \, (-2) \cdot \frac{du}{-2\pi} \quad + \int_1^0 2(u+1)(u^2) \quad du$$

$$= -\frac{1}{\pi} \cos u \Big|_{4\pi}^{2\pi} \quad + \int_1^0 2u^3 \, du \quad + \int_1^0 2u^2 \, du = 0 - \frac{2}{4} - \frac{2}{3}$$

$$= -\frac{7}{6}$$

Therefore if C is a curve then

$$\int_C P\,dx + Q\,dy = -\int_{-C} P\,dx + Q\,dy$$

Recall from last class if C is a curve then

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

Line integrals on Vector Fields

Two things needed :

- Vector Field

$$\vec{F}(x,y,z) = P(x,y,z)\,\hat{i} + Q(x,y,z)\,\hat{j} + R(x,y,z)\,\hat{k}$$

- Curve

$$C = \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$a \leq t \leq b$$

The line integral of $\vec{F}$ along C

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot \vec{r}\,'(t)\,dt$$

Example : Work done by a Force

If a Force $\vec{F}$, moves an object along a curve C

then work done is

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\, dt$$

Suppose a Force $\vec{F} = 8x^2yz\,\hat{i} + 5z\,\hat{j} - 4xy\,\hat{k}$

moved an object along C given by $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$,

$0 \le t \le 1$. Find the work done.

- Evaluate vector field along the curve

$$\vec{F}(\vec{r}(t)) = 8t^2 . t^2 . t^3\,\hat{i} + 5t^3\,\hat{j} - 4t . t^2\,\hat{k}$$
$$= 8t^7\,\hat{i} + 5t^3\,\hat{j} - 4t^3\,\hat{k}$$

$$\vec{r}\,'(t) = \frac{d}{dt}t\,\hat{i} + \frac{d}{dt}t^2\,\hat{j} + \frac{d}{dt}t^3\,\hat{k}$$
$$= 1\,\hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}$$

$$W = \int_{t=0}^{t=1} \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \left(8t^7\,\hat{i} + 5t^3\,\hat{j} - 4t^3\,\hat{k}\right) \bullet \left(1\,\hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}\right) dt$$

$$= \int_0^1 8t^7 + 10t^4 - 12t^5 \quad dt$$

$$= 8 \cdot \frac{t^8}{8}\Big|_0^1 + 10 \cdot \frac{t^5}{5}\Big|_0^1 - 12 \frac{t^6}{6}\Big|_0^1$$

$$= 1 + 2 - 2 = 1$$

Relation between Line Integral x, y, z and Line Integral Vector Field

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

where $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ and $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Proof: $\vec{F}(x, y, z) = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{k}$

$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$, $a \le t \le b$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \left(P\hat{\imath} + Q\hat{\jmath} + R\hat{k}\right) \cdot \left(x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}\right) dt$$

$$= \int_a^b P\,x'(t)\,dt + \int_a^b Q\,y'(t)\,dt + \int_a^b R\,z'(t)\,dt$$

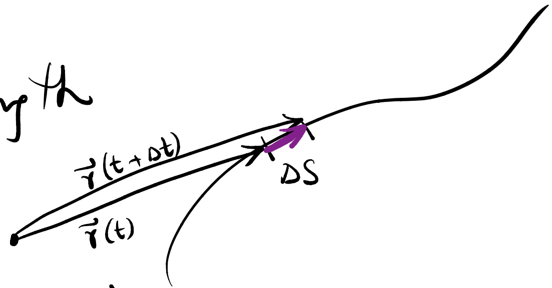
$$= \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz$$

Changing Orientation

$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_{C} \vec{F}\cdot dr$$

Question to think about

Recall along arc length



$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t) =$$

Therefore $ds = |\vec{r}\,'(t)|\, dt$

If $F(x,y,z) = \vec{F}\cdot \underbrace{\dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}_{\text{unit vector}}$

then $\displaystyle\int F(x,y,z)\, ds = \int \vec{F}\cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\, |r(t)|\, dt$

$$= \int \vec{F}\cdot \vec{r}\,'(t)\, dt = \int \vec{F}\cdot d\vec{r}$$

Why then

$$\int_C F\, ds = \int_{-C} F\, ds, \text{ but } \int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r} \quad ?$$