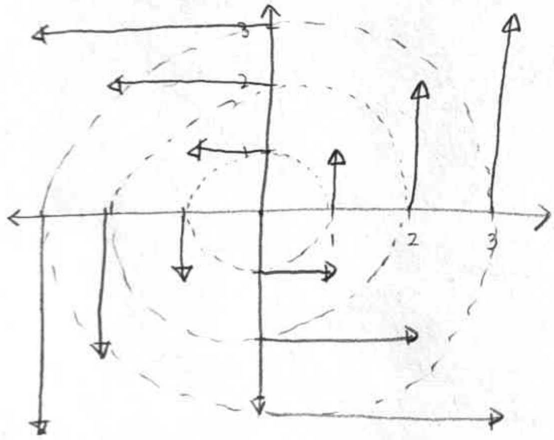
Lecture - 12

Vector - Fields

A vector field on $D \subseteq \mathbb{R}^2$ is a function $\vec{F}$ that assigns to each (x,y) in D a two dimensional vector $\vec{F}(x,y)$.

Similarly a vector field on $E \subseteq \mathbb{R}^3$ is a function $\vec{F}$ that assigns to each (x,y) in E a three dimensional vector $\vec{F}(x,y,z)$.

Eg: $\vec{F}(x,y) = -y\,\hat{i} + x\,\hat{j}$. Describe $\vec{F}$ by sketching $\vec{F}(x,y)$



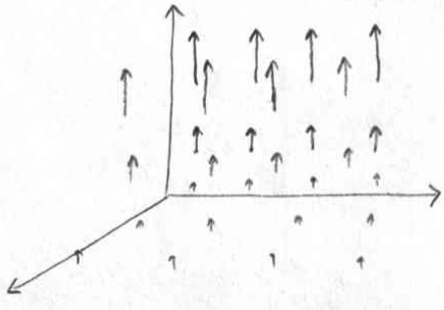
It seems each arrow at (x,y) is a tangent to the circle with center at origin.

To confirm $\vec{r} = x\hat{i} + y\hat{j}$

$\vec{F}(x,y) \quad -y\hat{i} + x\hat{j}$.

$$\therefore \quad \vec{r} \cdot \vec{F}(x,y) = -xy + xy = 0 .$$

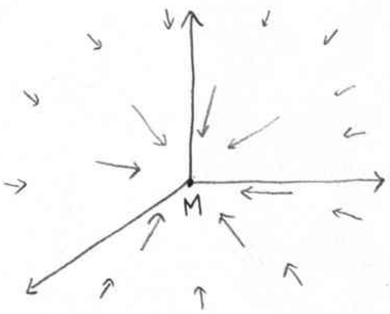
Eg Sketch vector field on $\mathbb{R}^3$ given by $\vec{F}(x,y,z) = z\,\hat{k}$



Eg Gravitational Field $(\vec{g})$

The force field is the Force (vector) experienced per unit mass in space.

Imagine a mass M at $(0,0,0)$



$$\vec{g}(x,y,z) = -\frac{GM \cdot 1}{|\vec{r}|^2} \cdot \hat{\vec{r}}$$

$$= -\frac{GM}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{GM\,\vec{r}}{|\vec{r}|^3}$$

Gradient as a vector field

Recall that if f is a scalar function of two variables then $\vec{\nabla} f$ is defined as

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath}$$

$$f_x = \frac{\partial f}{\partial x}$$

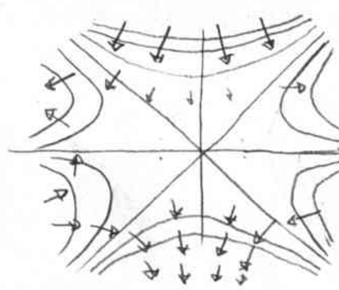
$$f_y = \frac{\partial f}{\partial y}$$

If f is a scalar function of three variables then $\vec{\nabla} f$ is:

$$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\,\hat{\imath} + f_y(x,y,z)\,\hat{\jmath} + f_z(x,y,z)\,\hat{k}.$$

Eg Find $\vec{\nabla} f$ for $f(x,y) = x^2y - y^3$. Plot gradient against contour map

$$\vec{\nabla} f = 2xy\,\hat{\imath} + (x^2 - 3y^2)\,\hat{\jmath}$$



Gradient vectors are perpendiculars to level curves